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Turbulent Boundary Layer in Compressible Fluids

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SUMMARY

The continuity, momentum, and energy differential equations for turbulent flow of a compressible fluid are derived, and the apparent turbulent stresses and dissipation function are identified. A general formula for skin friction, including heat transfer to a flat plate, is developed for a thin turbulent boundary layer in compressible fluids with zero pressure gradient. Curves are presented giving skin-friction coefficients and heat-transfer coefficients for air for various wall-to-free-stream temperature ratios and free-stream Mach Numbers.

In the special case when the boundary layer is insulated, this general formula yields skin-friction coefficients higher than those given by the von Kármán wall-property compressible-fluid formula but lower than those given by the von Kármán incompressible-fluid formula. Heat transfer from the boundary layer to the plate generally increases the friction and heat-transfer coefficients.

NOMENCLATURE

Variables

t = time
x = coordinate along plate in direction of free stream, measured from the leading edge
y = coordinate normal to plate, measured from the plate
z = coordinate along plate normal to free stream
u, v = x and y velocity components, respectively
p_x, p_y, p_z = normal stresses in the direction of x-, y-, and z-axes, respectively
τ = shear stress
τ_{yx} = shear stress in x-direction in plane normal to y-axis
τ_{xy} = shear stress in y-direction in plane normal to x-axis
q = heat transfer
q_x = heat transfer in x-direction
q_y = heat transfer in y-direction
ρ = fluid density
μ = coefficient of viscosity
k = coefficient of heat conductivity
T = absolute temperature
c_p = specific heat at constant pressure
ϵ = eddy coefficient of viscosity
κ = eddy coefficient of heat conductivity
γ = ratio of specific heats
M = Mach Number
l = mixing length
S = Sutherland constant
δ_c = thickness of the boundary layer when the fluid is compressible
δ_i = thickness of the boundary layer when the fluid is considered incompressible
C_f = total coefficient of friction
c_f = local coefficient of friction
R = Reynolds Number based on length x
N_{Pr} = Prandtl Number

Subscripts

w = wall condition
∞ = free-stream condition
lam. = laminar
turb. = turbulent

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INTRODUCTION

Two major problems encountered today in aeronautics are the determination of skin friction and skin temperatures of high-speed aircraft. Since the friction drag is a considerable portion of the total drag of a guided missile, it follows that miscalculation of the friction drag can result in considerable error in missile range. Furthermore, skin temperature is a decisive factor in the structural design of a high-speed missile.

These two problems arise as a result of the presence of the fluid boundary layer. Whether the boundary layer on a given missile is laminar or turbulent under certain conditions is as yet uncertain. However, in the present paper only the turbulent case is discussed. In particular, the purpose of the paper is to derive a general formula for skin friction including heat transfer to a flat plate for a thin fully turbulent boundary layer in compressible fluids with zero pressure gradient.

This work is a combination of two previous reports[1, 2] by the same author.

A first attempt to obtain theoretical results on turbulent skin friction on an insulated flat plate was made by von Kármán[3] when he suggested that the fluid state at the wall be used in the incompressible fluid formula; it seems that this suggestion is too optimistic because of the fact that there is a strong variation in temperature throughout the boundary layer. Frankl and Voishel[4] used von Kármán's similarity law for mixing length and, as a result of compressibility effects, encountered considerable complication in deriving an expression for turbulent friction with or without heat transfer. Ferrari[15] has made a general study of the problem; however, his apparent turbulent shear stress for compressible fluids is not in agreement with that of the author, and no engineering formula for skin friction or heat transfer is presented. Wilson[16] recently derived a formula for insulated plates using the von Kármán similarity law; his formula is similar to, but not the same as, the formula given in this paper.

In the present paper the differential equations for turbulent flow of a compressible fluid are derived for the purpose of identifying the apparent turbulent stresses and dissipation function. The boundary layer is then assumed thin, and the usual relation [Eq. (34)] between temperature and velocity is obtained when the turbulence Prandtl Number $c_p\epsilon/\kappa$ is taken as unity, where c_p is the specific heat at constant pressure, ϵ is the coefficient of eddy viscosity, and κ is the coefficient of eddy heat conductivity. The Prandtl wall differential equation is next derived for compressible fluids—viz.,

$$\frac{d\bar{u}}{dy} = \frac{1}{K}\sqrt{\frac{\tau_w}{\bar{\rho}}}\frac{1}{y}$$

Here, $\bar{u}$ is the mean local velocity, τ_w is the mean shear at the wall, $\bar{\rho}$ is the mean local fluid density, y is the distance from the wall, and K is the proportionality constant in the mixing length formula $l = Ky$. Finally, a general formula for skin friction including heat transfer is developed, and curves are presented giving skin-friction coefficients and heat-transfer coefficients for air for various wall-to-free-stream temperature ratios and free-stream Mach Numbers. For the insulated plate case, it is seen that the writer's results for skin friction lie above those of von Kármán. Cooling the wall generally increases the skin-friction and heat-transfer coefficients.

The Thin Turbulent Boundary Layer

The basic differential equations describing the two-dimensional flow in a boundary layer on a flat plate are

$$\text{Continuity:}\ \frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0 \tag{1}$$

Momentum:

$$\rho\frac{\partial u}{\partial t} + \rho u\frac{\partial u}{\partial x} + \rho v\frac{\partial u}{\partial y} = \frac{\partial p_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} \tag{2a}$$

$$\rho\frac{\partial v}{\partial t} + \rho u\frac{\partial v}{\partial x} + \rho v\frac{\partial v}{\partial y} = \frac{\partial p_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} \tag{2b}$$

Energy:

$$\rho\frac{\partial}{\partial t}(c_pT) + \rho u\frac{\partial}{\partial x}(c_pT) + \rho v\frac{\partial}{\partial y}(c_pT) - \frac{\partial p}{\partial t} - u\frac{\partial p}{\partial x} - v\frac{\partial p}{\partial y} = \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + (p+p_x)\frac{\partial u}{\partial x} + (p+p_y)\frac{\partial v}{\partial y} + \tau_{yx}\frac{\partial u}{\partial y} + \tau_{xy}\frac{\partial v}{\partial x} \tag{3}$$

where

$$p = -(1/3)(p_x + p_y + p_z) \tag{4}$$

$$p + p_x = -\frac{2}{3}\mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + 2\mu\frac{\partial u}{\partial x} \tag{5}$$

$$p + p_y = -\frac{2}{3}\mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + 2\mu\frac{\partial v}{\partial y} \tag{6}$$

$$p + p_z = -\frac{2}{3}\mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \tag{7}$$

$$\tau_{yx} = \tau_{xy} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \tag{8}$$

$$q_x = k\frac{\partial T}{\partial x},\quad q_y = k\frac{\partial T}{\partial y}$$

The last four terms on the right-hand side of Eq. (3) are called the dissipation function because they are functions of the viscosity.

The Mean Momentum Equations—Reynolds Stresses

The above momentum equations are rewritten in the following form by using the continuity equation:

$$\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(\rho u^2) + \frac{\partial}{\partial y}(\rho uv) = \frac{\partial p_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} \tag{9a}$$

$$\frac{\partial}{\partial t}(\rho v) + \frac{\partial}{\partial x}(\rho uv) + \frac{\partial}{\partial y}(\rho v^2) = \frac{\partial p_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} \tag{9b}$$

The following substitutions are then introduced into Eq. (9a):

$$u = \bar{u} + u',\quad \rho = \bar{\rho} + \rho',\quad \tau_{yx} = \overline{\tau_{yx}} + \tau_{yx}'$$

$$\rho u = \overline{\rho u} + (\rho u)',\quad \rho v = \overline{\rho v} + (\rho v)',\quad p_x = \overline{p_x} + p_x'$$

where the bars indicate slowly varying temporal mean values and the primes indicate instantaneous fluctuations from the mean. Time average of the resulting equation yields

$$\frac{\partial}{\partial t}(\bar{\rho}\cdot\bar{u} + \overline{\rho'u'}) + \frac{\partial}{\partial x}[\overline{\rho u}\cdot\bar{u} + \overline{(\rho u)'u'}] + \frac{\partial}{\partial y}[\overline{\rho v}\cdot\bar{u} + \overline{(\rho v)'u'}] = \frac{\partial \bar{p}_x}{\partial x} + \frac{\partial \overline{\tau_{yx}}}{\partial y} \quad (10)$$

By time average is meant, for example,

$$\bar{u} = \frac{1}{T}\int_{t-(T/2)}^{t+(T/2)} u\, dt$$

also, it can be shown that $\overline{\partial}/\partial = \partial^{-}/\partial$.

The equation of continuity, when averaged, becomes

$$\frac{\partial \bar{\rho}}{\partial t} + \frac{\partial}{\partial x}(\overline{\rho u}) + \frac{\partial}{\partial y}(\overline{\rho v}) = 0 \quad (11)$$

With some rearrangement and use of Eq. (11), Eq. (10) then reduces to

$$\bar{\rho}\frac{\partial \bar{u}}{\partial t} + \overline{\rho u}\frac{\partial \bar{u}}{\partial x} + \overline{\rho v}\frac{\partial \bar{u}}{\partial y} = \frac{\partial}{\partial t}(-\overline{\rho' u'}) + \frac{\partial}{\partial x}[\bar{p}_x - \overline{(\rho u)'u'}] + \frac{\partial}{\partial y}[\overline{\tau_{yx}} - \overline{(\rho v)'u'}] \quad (12a)$$

Comparison of Eqs. (12a) and (2a) shows that if, in Eq. (2a), ρ, u, ρu, ρv, p_x, and τ_{yx} are replaced by their mean values, then the terms $(\partial/\partial t)(-\overline{\rho'u'})$, $(\partial/\partial x) \times [-\overline{(\rho u)'u'}]$, and $(\partial/\partial y)[-\overline{(\rho v)'u'}]$ must be added to the right-hand side of the equation. The terms $-\overline{(\rho u)'u'}$ and $-\overline{(\rho v)'u'}$* will be called the Reynolds or apparent pressure and shear stresses, respectively, in the case of the turbulent flow of a compressible fluid. In the same manner, Eq. (9b) reduces to

$$\bar{\rho}\frac{\partial \bar{v}}{\partial t} + \overline{\rho v}\frac{\partial \bar{v}}{\partial x} + \overline{\rho v}\frac{\partial \bar{v}}{\partial y} = \frac{\partial}{\partial t}(\overline{\rho' v'}) + \frac{\partial}{\partial y}[\bar{p}_y - \overline{(\rho v)'v'}] + \frac{\partial}{\partial x}[\overline{\tau_{xy}} - \overline{(\rho u)'v'}] \quad (12b)$$

The boundary layer is now assumed to be thin—i.e., $\bar{v} << \bar{u}$. Hence, it is seen that terms on the left-hand side of Eq. (12b) are negligible compared to the corresponding terms in Eq. (12a), so that Eq. (12b) reduces to an equation devoid of mean motion terms. Now, in Eq. (12a), $\partial \bar{p}_x/\partial x$ is to be dropped because any external pressure gradient is omitted. $(\partial/\partial x)[-\overline{(\rho u)'u'}]$ should be negligible compared to $(\partial/\partial y)[-\overline{(\rho v)'u'}]$ because of thinness of the layer. Finally, $\partial\overline{\tau_{yx}}/\partial y$ definitely can be dropped because $\overline{\tau_{yx}}$ is the average of the viscous shear, analogous to incompressible fluid experience. The resulting momentum equation involving mean motion terms for thin turbulent boundary layers is, then,

$$\bar{\rho}\frac{\partial \bar{u}}{\partial t} + \overline{\rho u}\frac{\partial \bar{u}}{\partial x} + \overline{\rho v}\frac{\partial \bar{u}}{\partial y} = \frac{\partial}{\partial t}(-\overline{\rho' u'}) + \frac{\partial}{\partial y}[-\overline{(\rho v)'u'}] \quad (13)$$

and in the mean steady state

$$\overline{\rho u}\frac{\partial \bar{u}}{\partial x} + \overline{\rho v}\frac{\partial \bar{u}}{\partial y} = \frac{\partial}{\partial y}[-\overline{(\rho v)'u'}] \quad (14)$$

Now let

$$-\overline{(\rho v)'u'} = \epsilon(\partial \bar{u}/\partial y) \quad (15)$$

where ϵ shall be called the eddy viscosity as in incompressible fluid theory. Eq. (14) then becomes

$$\overline{\rho u}\frac{\partial \bar{u}}{\partial x} + \overline{\rho v}\frac{\partial \bar{u}}{\partial y} = \frac{\partial}{\partial y}\left(\epsilon\frac{\partial \bar{u}}{\partial y}\right) \quad (16)$$

* Expanded, this term becomes $-\overline{(\rho v)'u'} = -\bar{\rho}\overline{u'v'} - \bar{v}\overline{\rho'u'} - \overline{\rho'u'v'}$.

The Mean Energy Equation—Apparent Dissipation Function

Use of Eqs. (1) and (2) allows Eq. (3) to be written in the form

$$\frac{\partial}{\partial t}(\rho c_p T) + \frac{\partial}{\partial x}(\rho u c_p T) + \frac{\partial}{\partial y}(\rho v c_p T) + \frac{\partial}{\partial t}\left(\frac{\rho u^2}{2}\right) + \frac{\partial}{\partial x}\left(\frac{\rho u^3}{2}\right) + \frac{\partial}{\partial y}\left(\frac{\rho v u^2}{2}\right) + \frac{\partial}{\partial t}\left(\frac{\rho v^2}{2}\right) + \frac{\partial}{\partial x}\left(\frac{\rho u v^2}{2}\right) + \frac{\partial}{\partial y}\left(\frac{\rho v^3}{2}\right) - \frac{\partial p}{\partial t} - u\frac{\partial p}{\partial x} - v\frac{\partial p}{\partial y} = \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + u\frac{\partial \tau_{yx}}{\partial y} + u\frac{\partial p_x}{\partial x} + v\frac{\partial \tau_{xy}}{\partial x} + v\frac{\partial p_y}{\partial y} + (p + p_x)\frac{\partial y}{\partial x} + (p + p_y)\frac{\partial v}{\partial y} + \tau_{yx}\frac{\partial u}{\partial y} + \tau_{xy}\frac{\partial v}{\partial x} \quad (17)$$

As before, the following substitutions are then introduced:

$$\begin{array}{lll} u = \bar{u} + u' & T = \bar{T} + T' & p = \bar{p} + p' \\ v = \bar{v} + v' & \rho = \bar{\rho} + \rho' & p_x = \bar{p}_x + p_x' \\ \rho u = \overline{\rho u} + (\rho u)' & \tau_{yx} = \overline{\tau_{yx}} + \tau_{yx}' & p_y = \bar{p}_y + p_y' \\ \rho v = \overline{\rho v} + (\rho v)' & \tau_{xy} = \overline{\tau_{xy}} + \tau_{xy}' & q_y = \bar{q}_y + q_y' \\ & & q_x = \bar{q}_x + q_x' \end{array}$$

where the mean values again are slowly varying. With these substitutions inserted, Eq. (17) becomes

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left[c_p(\bar\rho + \rho')(\bar T + T')\right] + \frac{\partial}{\partial x}\left\{c_p\left[\overline{\rho u} + (\rho u)'\right](\bar T + T')\right\} + \frac{\partial}{\partial y}\left\{c_p\left[\overline{\rho v} + (\rho v)'\right](\bar T + T')\right\} + \\
&\frac{\partial}{\partial t}\left[\frac{1}{2}(\bar\rho + \rho')(\bar u + u')^2\right] + \frac{\partial}{\partial x}\left\{\frac{1}{2}\left[\overline{\rho u} + (\rho u)'\right](\bar u + u')^2\right\} + \frac{\partial}{\partial y}\left\{\frac{1}{2}\left[\overline{\rho v} + (\rho v)'\right](\bar u + u')^2\right\} + \\
&\frac{\partial}{\partial t}\left[\frac{1}{2}(\bar\rho + \rho')(\bar v + v')^2\right] + \frac{\partial}{\partial x}\left\{\frac{1}{2}\left[\overline{\rho u} + (\rho u)'\right](\bar v + v')^2\right\} + \frac{\partial}{\partial y}\left\{\frac{1}{2}\left[\overline{\rho v} + (\rho v)'\right](\bar v + v')^2\right\} = \\
&\frac{\partial}{\partial t}(\bar p + p') + (\bar u + u')\frac{\partial}{\partial x}(\bar p + p') + (\bar v + v')\frac{\partial}{\partial y}(\bar p + p') + \frac{\partial}{\partial x}(\bar q_x + q_x') + \frac{\partial}{\partial y}(\bar q_y + q_y') + \\
&(\bar u + u')\frac{\partial}{\partial y}(\bar\tau_{yx} + \tau_{yx}') + (\bar u + u')\frac{\partial}{\partial x}(\bar p_x + p_x') + (\bar v + v')\frac{\partial}{\partial x}(\bar\tau_{xy} + \tau_{xy}') + (\bar v + v')\frac{\partial}{\partial y}(\bar p_y + p_y') + \\
&(\bar p + p' + \bar p_x + p_x')\frac{\partial}{\partial x}(\bar u + u') + (\bar p + p' + \bar p_y + p_y')\frac{\partial}{\partial y}(\bar v + v') + (\bar\tau_{yx} + \tau_{yx}')\frac{\partial}{\partial y}(\bar u + u') + \\
&(\bar\tau_{xy} + \tau_{xy}')\frac{\partial}{\partial x}(\bar v + v') = \frac{\partial}{\partial t}(\bar p + p') + \frac{\partial}{\partial x}\left\{\left[(\bar p + p') + (\bar p_x + p_x')\right](\bar u + u')\right\} + \\
&\frac{\partial}{\partial y}\left\{\left[(\bar p + p')(\bar p_y + p_y')\right](\bar v + v')\right\} + \frac{\partial}{\partial x}(\bar q_x + q_x') + \frac{\partial}{\partial y}(\bar q_y + q_y') + \frac{\partial}{\partial y}\left[(\bar\tau_{yx} + \tau_{yx}')(\bar u + u')\right] + \\
&\frac{\partial}{\partial x}\left[(\bar\tau_{xy} + \tau_{xy}')(\bar v + v')\right] \qquad (18)
\end{aligned}
$$

Upon multiplication of the terms in the brackets and averaging term by term over time, there results

$$
\begin{aligned}
&\frac{\partial}{\partial t}(c_p\bar\rho\cdot\bar T + c_p\overline{\rho' T'}) + \frac{\partial}{\partial x}\left[c_p\overline{\rho u}\cdot\bar T + c_p\overline{(\rho u)'T'}\right] + \frac{\partial}{\partial y}\left[c_p\overline{\rho v}\cdot\bar T + c_p\overline{(\rho v)'T'}\right] + \\
&\frac{1}{2}\frac{\partial}{\partial t}(\bar\rho\cdot\bar u^2 + \bar\rho\cdot\overline{u'^2} + 2\bar u\cdot\overline{\rho' u'} + \overline{\rho' u'^2}) + \frac{1}{2}\frac{\partial}{\partial x}\left[\overline{\rho u}\cdot\bar u^2 + \overline{\rho u}\cdot\overline{u'^2} + 2\bar u\cdot\overline{(\rho u)'u'} + \overline{(\rho u)'u'^2}\right] + \\
&\frac{1}{2}\frac{\partial}{\partial y}\left[\overline{\rho v}\cdot\bar u^2 + \overline{\rho v}\cdot\overline{u'^2} + 2\bar u\,\overline{(\rho v)'u'} + \overline{(\rho v)'u'^2}\right] + \frac{1}{2}\frac{\partial}{\partial t}\left[\bar\rho\cdot\bar v^2 + \bar\rho\cdot\overline{v'^2} + 2\bar v\cdot\overline{\rho' v'} + \overline{\rho' v'^2}\right] + \\
&\frac{1}{2}\frac{\partial}{\partial x}\left[\overline{\rho u}\cdot\bar v^2 + \overline{\rho u}\cdot\overline{v'^2} + 2\bar v\cdot\overline{(\rho u)'v'} + \overline{(\rho u)'v'^2}\right] + \frac{1}{2}\frac{\partial}{\partial y}\left[\overline{\rho v}\cdot\bar v^2 + \overline{\rho v}\cdot\overline{v'^2} + 2\bar v\,\overline{(\rho v)'v'} + \overline{(\rho v)'v'^2}\right] = \\
&\frac{\partial\bar p}{\partial t} + \frac{\partial}{\partial x}(\bar p\cdot\bar u + \bar p_x\cdot\bar u + \overline{p'u'} + \overline{p_x'u'}) + \frac{\partial}{\partial y}(\bar p\cdot\bar v + \bar p_y\bar v + \overline{p'v'} + \overline{p_y'v'}) + \frac{\partial\bar q_x}{\partial x} + \frac{\partial\bar q_y}{\partial y} + \\
&\frac{\partial}{\partial y}(\bar\tau_{yx}\cdot\bar u + \overline{\tau_{yx}'u'}) + \frac{\partial}{\partial x}(\bar\tau_{xy}\cdot\bar v + \overline{\tau_{xy}'v'}) \qquad (19)
\end{aligned}
$$

Carrying out the indicated differentiation of Eq. (19), one obtains

$$
\begin{aligned}
&c_p\bar T\cdot\frac{\partial\bar\rho}{\partial t} + \bar\rho\frac{\partial}{\partial t}(c_p\bar T) + \frac{\partial}{\partial t}(c_p\overline{\rho' T'}) + c_p\bar T\frac{\partial}{\partial x}(\overline{\rho u}) + \overline{\rho u}\frac{\partial}{\partial x}(c_p\bar T) + \frac{\partial}{\partial x}\left[c_p\overline{(\rho u)'T'}\right] + c_p\bar T\frac{\partial}{\partial y}(\overline{\rho v}) + \\
&\overline{\rho v}\frac{\partial}{\partial y}(c_p\bar T) + \frac{\partial}{\partial y}\left[c_p\overline{(\rho v)'T'}\right] + \bar\rho\cdot\bar u\cdot\frac{\partial\bar u}{\partial t} + \frac{\bar u^2}{2}\frac{\partial\bar\rho}{\partial t} + \bar\rho\frac{\partial}{\partial t}\left(\frac{\overline{u'^2}}{2}\right) + \frac{\overline{u'^2}}{2}\frac{\partial\bar\rho}{\partial t} + \bar u\frac{\partial}{\partial t}(\overline{\rho' u'}) + \overline{\rho' u'}\frac{\partial\bar u}{\partial t} + \\
&\frac{\partial}{\partial t}\left(\frac{\overline{\rho' u'^2}}{2}\right) + \bar u\cdot\overline{\rho u}\cdot\frac{\partial\bar u}{\partial x} + \frac{\bar u^2}{2}\frac{\partial}{\partial x}(\overline{\rho u}) + \overline{\rho u}\frac{\partial}{\partial x}\left(\frac{\overline{u'^2}}{2}\right) + \frac{\overline{u'^2}}{2}\frac{\partial}{\partial x}(\overline{\rho u}) + \bar u\frac{\partial}{\partial x}\left[\overline{(\rho u)'u'}\right] + \overline{(\rho u)'u'}\cdot\frac{\partial\bar u}{\partial x} + \\
&\frac{\partial}{\partial x}\left[\frac{\overline{(\rho u)'u'^2}}{2}\right] + \bar u\cdot\overline{\rho v}\frac{\partial\bar u}{\partial y} + \frac{\bar u^2}{2}\frac{\partial}{\partial y}(\overline{\rho v}) + \overline{\rho v}\frac{\partial}{\partial y}\left(\frac{\overline{u'^2}}{2}\right) + \frac{\overline{u'^2}}{2}\frac{\partial}{\partial y}(\overline{\rho v}) + \bar u\frac{\partial}{\partial y}\left[\overline{(\rho v)'u'}\right] + \overline{(\rho v)'u'}\frac{\partial\bar u}{\partial y} + \\
&\frac{\partial}{\partial y}\left[\frac{\overline{(\rho v)'u'^2}}{2}\right] + \bar\rho\cdot\bar v\frac{\partial\bar v}{\partial t} + \frac{\bar v^2}{2}\frac{\partial\bar\rho}{\partial t} + \bar\rho\frac{\partial}{\partial t}\left(\frac{\overline{v'^2}}{2}\right) + \frac{\overline{v'^2}}{2}\cdot\frac{\partial\bar\rho}{\partial t} + \bar v\frac{\partial}{\partial t}(\overline{\rho' v'}) + \overline{\rho' v'}\cdot\frac{\partial\bar v}{\partial t} + \frac{\partial}{\partial t}\left(\frac{\overline{\rho' v'^2}}{2}\right) + \bar v\,(\overline{\rho u})\cdot\frac{\partial\bar v}{\partial x} + \\
&\frac{\bar v^2}{2}\frac{\partial}{\partial x}(\overline{\rho u}) + \overline{\rho u}\frac{\partial}{\partial x}\left(\frac{\overline{v'^2}}{2}\right) + \frac{\overline{v'^2}}{2}\cdot\frac{\partial}{\partial x}(\overline{\rho u}) + \bar v\frac{\partial}{\partial x}\left[\overline{(\rho u)'v'}\right] + \overline{(\rho u)'v'}\frac{\partial\bar v}{\partial x} + \frac{\partial}{\partial x}\left[\frac{\overline{(\rho u)'v'^2}}{2}\right] + \bar v\,(\overline{\rho v})\cdot\frac{\partial\bar v}{\partial y} + \\
&\frac{\bar v^2}{2}\cdot\frac{\partial}{\partial y}(\overline{\rho v}) + \overline{\rho v}\frac{\partial}{\partial y}\left(\frac{\overline{v'^2}}{2}\right) + \frac{\overline{v'^2}}{2}\frac{\partial}{\partial y}(\overline{\rho v}) + \bar v\frac{\partial}{\partial y}\left[\overline{(\rho v)'v'}\right] + \overline{(\rho v)'v'}\frac{\partial\bar v}{\partial y} + \frac{\partial}{\partial y}\left[\frac{\overline{(\rho v)'v'^2}}{2}\right] = \frac{\partial\bar p}{\partial t} + \bar p\frac{\partial\bar u}{\partial x} + \\
&\bar u\frac{\partial\bar p}{\partial x} + \bar p_x\frac{\partial\bar u}{\partial x} + \bar u\frac{\partial\bar p_x}{\partial x} + \frac{\partial}{\partial x}(\overline{p'u'}) + \frac{\partial}{\partial x}(\overline{p_x'u'}) + \bar p\frac{\partial\bar v}{\partial y} + \bar v\frac{\partial\bar p}{\partial y} + \bar p_y\frac{\partial\bar v}{\partial y} + \bar v\frac{\partial\bar p_y}{\partial y} + \frac{\partial}{\partial y}(\overline{p'v'}) + \\
&\frac{\partial}{\partial y}(\overline{p_y'v'}) + \frac{\partial\bar q_x}{\partial x} + \frac{\partial\bar q_y}{\partial y} + \bar\tau_{yx}\cdot\frac{\partial\bar u}{\partial y} + \bar u\cdot\frac{\partial\bar\tau_{yx}}{\partial y} + \frac{\partial}{\partial y}(\overline{\tau_{yx}'u'}) + \bar\tau_{xy}\cdot\frac{\partial\bar v}{\partial x} + \bar v\cdot\frac{\partial\bar\tau_{xy}}{\partial x} + \frac{\partial}{\partial x}(\overline{\tau_{xy}'v'}) \qquad (20)
\end{aligned}
$$

This equation can be reduced by use of the continuity and momentum equations. The continuity equation [Eq. (11)] eliminates the 1st, 4th, 7th, 11th, 13th, 18th, 20th, 25th, 27th, 32nd, 34th, 39th, 41st, 46th, and 48th terms on the left-hand side. The momentum equations [Eqs. (12a) and (12b)] eliminate the 10th, 14th, 17th, 21st, 24th, 28th, 31st, 35th, 38th, 42nd, 45th, and 49th terms on the left-hand side and the 5th, 11th, 17th, and 20th terms on the right-hand side of the equation. Collecting the remaining terms then gives

$$\begin{aligned}\bar{\rho}\frac{\partial}{\partial t}(c_p\bar{T}) + \overline{\rho u}\frac{\partial}{\partial x}(c_p\bar{T}) + \overline{\rho v}\frac{\partial}{\partial y}(c_p\bar{T}) - \frac{\partial\bar{p}}{\partial t} - \bar{u}\frac{\partial\bar{p}}{\partial x} - \bar{v}\frac{\partial\bar{p}}{\partial y} = \frac{\partial}{\partial t}(-c_p\overline{\rho' T'}) - \overline{\rho' u'}\frac{\partial\bar{u}}{\partial t} - \overline{\rho' v'}\frac{\partial\bar{v}}{\partial t} - \\ \bar{\rho}\frac{\partial}{\partial t}\left(\frac{\overline{u'^2}}{2} + \frac{\overline{v'^2}}{2}\right) - \overline{\rho u}\frac{\partial}{\partial x}\left(\frac{\overline{u'^2}}{2} + \frac{\overline{v'^2}}{2}\right) - \overline{\rho v}\frac{\partial}{\partial y}\left(\frac{\overline{u'^2}}{2} + \frac{\overline{v'^2}}{2}\right) - \frac{\partial}{\partial t}\left(\frac{1}{2}\overline{\rho' u'^2} + \frac{1}{2}\overline{\rho' v'^2}\right) + \\ \frac{\partial}{\partial x}(\overline{p'u'} + \overline{p_x' u'}) - \frac{\partial}{\partial x}\left[\frac{\overline{(\rho u)' u'^2}}{2}\right] + \frac{\partial}{\partial y}(\overline{p'v'} + \overline{p_y' v'}) - \frac{\partial}{\partial y}\left[\frac{\overline{(\rho v)' v'^2}}{2}\right] + \frac{\partial}{\partial x}(\overline{\tau_{xy}' v'}) - \frac{\partial}{\partial x}\left[\frac{\overline{(\rho u)' v'^2}}{2}\right] + \\ \frac{\partial}{\partial y}(\overline{\tau_{yx}' u'}) - \frac{\partial}{\partial y}\left[\frac{\overline{(\rho v)' u'^2}}{2}\right] + \frac{\partial\bar{q}_x}{\partial x} - \frac{\partial}{\partial x}[c_p\overline{(\rho u)' T'}] + \frac{\partial\bar{q}_y}{\partial y} - \frac{\partial}{\partial y}[c_p\overline{(\rho v)' T'}] + (\bar{p} + \bar{p}_x)\frac{\partial\bar{u}}{\partial x} - \\ \overline{(\rho u)' u'}\frac{\partial\bar{u}}{\partial x} + (\bar{p} + \bar{p}_y)\frac{\partial\bar{v}}{\partial y} - \overline{(\rho v)' v'}\frac{\partial\bar{v}}{\partial y} + \bar{\tau}_{yx}\frac{\partial\bar{u}}{\partial y} - \overline{(\rho v)' u'}\frac{\partial\bar{u}}{\partial y} + \bar{\tau}_{xy}\cdot\frac{\partial\bar{v}}{\partial x} - \overline{(\rho u)' v'}\frac{\partial\bar{v}}{\partial x} \quad (21)\end{aligned}$$

Comparison of Eqs. (21) and (3) shows what terms must be added when the instantaneous values of the variables are to be replaced by their mean values. The last eight terms of Eq. (21) correspond to the last four of Eq. (3); hence, the additive terms

$$-\overline{(\rho u)' u'}\,(\partial\bar{u}/\partial x), \quad -\overline{(\rho v)' v'}\,(\partial\bar{v}/\partial y), \quad -\overline{(\rho v)' u'}\,(\partial\bar{u}/\partial y), \quad -\overline{(\rho u)' v'}\,(\partial\bar{v}/\partial x)$$

will be called apparent dissipation function terms.

Eq. (21) can be reduced further. (1) In Eq. (19), $\overline{u'^2}$ and $\overline{v'^2}$ are expected to be small compared to $\bar{u}^2$, and, since $\bar{u}^2$ is assumed of the same order as $c_p\bar{T}$, they can be dropped from the equation. (2) The triple correlations are also expected to be small and, hence, can be neglected. Next, because of thinness of boundary layer and absence of external pressure gradient, terms containing pressure gradients should be negligible. Hence, Eq. (21) becomes

$$\begin{aligned}\bar{\rho}\frac{\partial}{\partial t}(c_p\bar{T}) + \overline{\rho u}\frac{\partial}{\partial x}(c_p\bar{T}) + \overline{\rho v}\frac{\partial}{\partial y}(c_p\bar{T}) - \frac{\partial\bar{p}}{\partial t} = \frac{\partial}{\partial t}(-c_p\,\overline{\rho' T'}) - \overline{\rho' u'}\frac{\partial\bar{u}}{\partial t} - \overline{\rho' v'}\frac{\partial\bar{v}}{\partial t} + \\ \frac{\partial}{\partial x}(\overline{\tau_{xy}' v'}) + \frac{\partial}{\partial y}(\overline{\tau_{yx}' u'}) + \frac{\partial\bar{q}_x}{\partial x} + \frac{\partial}{\partial x}[-c_p\,\overline{(\rho u)' T'}] + \frac{\partial\bar{q}_y}{\partial y} + \frac{\partial}{\partial y}[-c_p\overline{(\rho v)' T'}] + (\bar{p} + \bar{p}_x)\frac{\partial\bar{u}}{\partial x} - \\ \overline{(\rho u)' u'}\cdot\frac{\partial\bar{u}}{\partial x} + (\bar{p} + \bar{p}_y)\frac{\partial\bar{v}}{\partial y} - \overline{(\rho v)' v'}\cdot\frac{\partial\bar{v}}{\partial y} + \bar{\tau}_{yx}\frac{\partial\bar{u}}{\partial y} - \overline{(\rho v)' u'}\cdot\frac{\partial\bar{u}}{\partial y} + \bar{\tau}_{xy}\frac{\partial\bar{v}}{\partial y} - \overline{(\rho u)' v'}\cdot\frac{\partial\bar{v}}{\partial x} \quad (22)\end{aligned}$$

Terms representing molecular action can further be neglected compared to terms representing molar action; hence, the 4th, 5th, 6th, 8th, 14th, and 16th terms on the right-hand side can be ignored. Because of thinness of the layer, $(\partial/\partial x)\ [-c_p\overline{(\rho u)' T'}]$ should be small relative to $(\partial/\partial y)\ [-c_p\overline{(\rho v)' T'}]$ and can be dropped. Furthermore, $\partial\bar{u}/\partial x$, $\partial\bar{v}/\partial y$, and $\partial\bar{v}/\partial x$ are much smaller than $\partial\bar{u}/\partial y$, and, hence, all remaining terms containing these as factors will be assumed small and will be ignored. Also, $-\overline{\rho' v'}\ (\partial\bar{v}/\partial t)$ is dropped, since it is expected to be small relative to $-\overline{\rho' u'}\ (\partial\bar{u}/\partial t)$ because $\bar{v} \ll \bar{u}$. As a final approximate result for thin turbulent boundary layers, Eq. (20) reduces to

$$\begin{aligned}\bar{\rho}\frac{\partial}{\partial t}(c_p\bar{T}) + \overline{\rho u}\frac{\partial}{\partial x}(c_p\bar{T}) + \overline{\rho v}\frac{\partial}{\partial y}(c_p\bar{T}) - \frac{\partial\bar{p}}{\partial t} = \\ -\overline{\rho' u'}\frac{\partial\bar{u}}{\partial t} + \frac{\partial}{\partial t}(-c_p\overline{\rho' T'}) + \frac{\partial}{\partial y}[-c_p\overline{(\rho v)' T'}] - \\ \overline{(\rho v)' u'}\frac{\partial\bar{u}}{\partial y} \quad (23)\end{aligned}$$

and in the mean steady state

$$\overline{\rho u}\frac{\partial}{\partial x}(c_p\bar{T}) + \overline{\rho v}\frac{\partial}{\partial y}(c_p\bar{T}) = \frac{\partial}{\partial y}[-c_p\overline{(\rho v)' T'}] - \overline{(\rho v)' u'}\frac{\partial\bar{u}}{\partial y} \quad (24)$$

The following definition is now introduced—viz.:

$$-c_p\overline{(\rho v)' T'} = \kappa(\partial\bar{T}/\partial y) \quad (25)$$

where κ will be called the eddy heat conductivity as in incompressible fluid theory. Hence, with Eqs. (15) and (25), Eq. (24) becomes

$$\overline{\rho u}\frac{\partial}{\partial x}(c_p\bar{T}) + \overline{\rho v}\frac{\partial}{\partial y}(c_p\bar{T}) = \frac{\partial}{\partial y}\left(\kappa\frac{\partial\bar{T}}{\partial y}\right) + \epsilon\left(\frac{\partial\bar{u}}{\partial y}\right)^2 \quad (26)$$

A Solution of the Energy Equation

The mean steady turbulent-flow equations for continuity, momentum, and energy for thin boundary layers are collected

$$(\partial/\partial x)\,(\overline{\rho u}) + (\partial/\partial y)\,(\overline{\rho v}) = 0 \qquad (27a)$$

$$\overline{\rho u}\cdot\frac{\partial \bar{u}}{\partial x} + \overline{\rho v}\,\frac{\partial \bar{u}}{\partial y} = \frac{\partial}{\partial y}\left(\epsilon\,\frac{\partial \bar{u}}{\partial y}\right) \qquad (27b)$$

$$\overline{\rho v}\cdot\frac{\partial}{\partial x}\cdot(c_p\bar{T}) + \overline{\rho v}\,\frac{\partial}{\partial y}\,(c_p\bar{T}) = \frac{\partial}{\partial y}\left(\kappa\,\frac{\partial \bar{T}}{\partial y}\right) + \epsilon\left(\frac{\partial \bar{u}}{\partial y}\right)^2 \qquad (27c)$$

Now, analogous to the procedure with laminar boundary layers, where the Prandtl Number $c_p\mu/k$ is taken as unity because its actual value for air is near unity, a turbulence Prandtl Number will be invented which will be taken equal to unity. Thus, it is assumed that

$$c_p\epsilon/\kappa = 1 \qquad (28)$$

This assumption seems justified by the fact that ϵ and κ result from the same internal mechanism in turbulent flow just as μ and k result from the same internal mechanism in laminar gaseous flow. Hence, since c_p is constant, Eq. (27c) becomes

$$\overline{\rho u}\,\frac{\partial}{\partial x}\,(c_p\bar{T}) + \overline{\rho v}\,\frac{\partial}{\partial y}\,(c_p\bar{T}) = \frac{\partial}{\partial y}\left[\epsilon\,\frac{\partial}{\partial y}\,(c_p\bar{T})\right] + \epsilon\left(\frac{\partial \bar{u}}{\partial y}\right)^2 \qquad (29)$$

The temperature is now assumed to be a function of the x-component of velocity—i.e., $c_p\bar{T} = f(\bar{u})$. Therefore, Eq. (29) becomes

$$\overline{\rho u}\,\frac{\partial \bar{u}}{\partial x}\,\frac{df(\bar{u})}{d\bar{u}} + \overline{\rho v}\,\frac{\partial \bar{u}}{\partial y}\,\frac{df(\bar{u})}{d\bar{u}} = \frac{\partial}{\partial y}\left[\epsilon\,\frac{\partial \bar{u}}{\partial y}\,\frac{df(\bar{u})}{d\bar{u}}\right] + \epsilon\left(\frac{\partial \bar{u}}{\partial y}\right)^2 = \frac{df(\bar{u})}{d\bar{u}}\,\frac{\partial}{\partial y}\left(\epsilon\,\frac{\partial \bar{u}}{\partial y}\right) + \epsilon\left(\frac{\partial \bar{u}}{\partial y}\right)^2 \frac{d^2 f(\bar{u})}{d\bar{u}^2} + \epsilon\left(\frac{\partial \bar{u}}{\partial y}\right)^2 = \frac{df(\bar{u})}{d\bar{u}}\,\frac{\partial}{\partial y}\left(\epsilon\,\frac{\partial \bar{u}}{\partial y}\right) + \epsilon\left[\frac{d^2 f(\bar{u})}{d\bar{u}^2} + 1\right]\left(\frac{\partial \bar{u}}{\partial y}\right)^2 \qquad (30)$$

Hence, the momentum Eq. (27b) is satisfied if

$$d^2f(\bar{u})/d\bar{u}^2 = -1 \qquad (31)$$

which upon integration gives

$$c_p\bar{T} = A_1 + B_1\bar{u} - (\bar{u}^2/2) \qquad (32)$$

where A_1 and B_1 are constants of integration.

The Prandtl Number of the laminar sublayer is next assumed equal to unity. Since in the case of laminar flow an equation identical to Eq. (32) is obtained for Prandtl Number equal to unity,[5] it follows that Eq. (32) may be used across the laminar sublayer to the wall. Hence, with the boundary conditions

$$\bar{T} = T_w \text{ at } \bar{u} = 0$$
$$\bar{T} = T_\infty \text{ at } \bar{u} = u_\infty$$

and the fact that

$$u_\infty{}^2/2c_pT_\infty = [(\gamma - 1)/2]M_\infty{}^2 \qquad (33)$$

Eq. (32) becomes

$$\frac{\bar{T}}{T_\infty} = \frac{T_w}{T_\infty} - \left(\frac{T_w}{T_\infty} - 1\right)\frac{\bar{u}}{\bar{u}_\infty} + \frac{\gamma - 1}{2}\,M_\infty{}^2\,\frac{\bar{u}}{u_\infty}\left(1 - \frac{\bar{u}}{u_\infty}\right) \qquad (34)$$

where the subscripts ∞ and w refer to the free stream and wall, respectively; γ is the ratio of the specific heats; and M_∞ is the ratio of the free-stream velocity to the velocity of sound in the free stream. (Crocco[6] has previously obtained this equation.)

Differentiation of Eq. (34) yields

$$\frac{1}{T_\infty}\left(\frac{\partial \bar{T}}{\partial y}\right) = \frac{1}{u_\infty}\left[\left(1 - \frac{T_w}{T_\infty}\right) + \frac{\gamma - 1}{2}\,M_\infty{}^2\right]\frac{\partial \bar{u}}{\partial y} - (\gamma - 1)M_\infty{}^2\,\frac{\bar{u}}{u_\infty{}^2}\,\frac{\partial \bar{u}}{\partial y}$$

which becomes, at the wall,

$$\left(\frac{\partial \bar{T}}{\partial y}\right)_w = \frac{T_\infty}{u_\infty}\left[\left(1 - \frac{T_w}{T_\infty}\right) + \frac{\gamma - 1}{2}\,M_\infty{}^2\right]\left(\frac{\partial \bar{u}}{\partial y}\right)_w \qquad (35)$$

or, introducing the coefficient of heat conductivity at the wall, k_w, and the coefficient of viscosity at the wall, μ_w,

$$k_w\left(\frac{\partial \bar{T}}{\partial y}\right)_w = \frac{k_w}{\mu_w}\cdot\frac{T_\infty}{u_\infty}\left[\left(1 - \frac{T_w}{T_\infty}\right) + \frac{\gamma - 1}{2}\,M_\infty{}^2\right]\mu_w\left(\frac{\partial \bar{u}}{\partial y}\right)_w$$

whence, by definition of q and τ and the assumption that $c_p\mu_w/k_w = 1$,

$$\frac{q_w}{\tau_w} = c_p\,\frac{T_\infty}{u_\infty}\left[\left(1 - \frac{T_w}{T_\infty}\right) + \frac{\gamma - 1}{2}\,M_\infty{}^2\right] \qquad (36)$$

valid for turbulent, as well as laminar, flow.

In the special case when the boundary layer is insulated, $q_w = 0$ and Eq. (36) yields

$$T_w/T_\infty = 1 + [(\gamma - 1)/2]M_\infty{}^2 \qquad (37)$$

which, when substituted into Eq. (34), gives

$$\frac{\bar{T}}{T_\infty} = \frac{T_w}{T_\infty} - \frac{\gamma - 1}{2}\,M_\infty{}^2\left(\frac{\bar{u}}{u_\infty}\right)^2 \qquad (38)$$

Use of Eq. (33) then gives

$$\bar{T}/T_\infty = (T_w/T_\infty) - (\bar{u}^2/2c_pT_\infty)$$

or

$$c_pT_w = c_p\bar{T} + (\bar{u}^2/2) = \text{constant} \qquad (39)$$

Eq. (39) states that, in the case of an insulated flat plate, the energy content per unit mass is constant across the boundary layer whether laminar or turbulent.

In the general case of heat transfer to or from the boundary layer, Eq. (36) can now be rewritten, with the aid of Eq. (37), in the form

$$q_w = (c_p\tau_w/u_\infty)\,(T_{w_{ins.}} - T_w) \tag{40}$$

where $T_{w_{ins.}}$ is the temperature the wall would acquire if the boundary layer were insulated.

The film coefficient of heat transfer, h, will be defined by

$$q_w = h(T_{w_{ins.}} - T_w) \tag{41}$$

so that

$$h = c_p\,(\tau_w/u_\infty) \tag{42}$$

Defining a dimensionless heat-transfer coefficient by

$$C_{H_\infty} = h/c_p\rho_\infty u_\infty \tag{43}$$

and the local coefficient of friction by

$$c_{f_\infty} = 2\tau_w/\rho_\infty u_\infty^2 \tag{44}$$

it follows that

$$C_{H_\infty} = (1/2)c_{f_\infty} \tag{45}$$

Therefore, once the local coefficient of friction is known, the local coefficient of heat transfer is directly obtained. Any refinement in the above procedure would require further knowledge of the turbulence mechanism and the extent of the laminar sublayer.

Turbulent Skin Friction with Heat Transfer

It has been shown in a previous section that the turbulent shear stress is given by

$$\begin{aligned}\tau &= -\overline{(\rho v)'u'}\\ &= -\bar{\rho}\,\overline{u'v'} - \bar{v}\,\overline{\rho'u'} - \overline{\rho'u'v'}\end{aligned}$$

In the case of the thin boundary layer where $v \approx 0$, this equation yields

$$\tau = -\bar{\rho}\,\overline{u'v'} - \overline{\rho'u'v'} \tag{46}$$

Now, if the triple correlation is neglected, Eq. (46) reduces further to

$$\tau = -\bar{\rho}\,\overline{u'v'} \tag{47}$$

which has the same form as for incompressible fluids. Introduction of the Prandtl mixing length, l, then gives

$$\tau = \bar{\rho}l^2(d\bar{u}/dy)^2$$

Following Prandtl's method, a wall formula is next obtained by assuming that $l = Ky$ and that the shear stress is constant near the wall; thus,

$$\tau_w = \bar{\rho}K^2y^2(dy/d\bar{u})^2$$

or, rearranging,

$$d\bar{u}/dy = (1/K)\,\sqrt{\tau_w/\bar{\rho}}\,(1/y) \tag{48}$$

which is the same as the Prandtl incompressible fluid wall formula, except that the density is variable.

Since the pressure is constant through the thin boundary layer, there results from the perfect-gas law

$$\bar{\rho}/\rho_w = T_w/\bar{T} \tag{49}$$

Now, rearrangement of Eq. (34) yields

$$\frac{\bar{T}}{T_w} = 1 + \left[\left(1 + \frac{\gamma - 1}{2}M_\infty^2\right)\frac{T_\infty}{T_w} - 1\right]\frac{\bar{u}}{u_\infty} - \frac{\gamma - 1}{2}M_\infty^2\frac{T_\infty}{T_w}\left(\frac{\bar{u}}{u_\infty}\right)^2 \tag{50}$$

so that from Eq. (49)

$$\frac{\bar{\rho}}{\rho_w} = \frac{1}{1 + B\,(\bar{u}/u_\infty) - A^2(\bar{u}/u_\infty)^2} \tag{51}$$

where

$$A^2 = \frac{\dfrac{\gamma - 1}{2}M_\infty^2}{T_w/T_\infty} \text{ and } B = \frac{1 + \dfrac{\gamma - 1}{2}M_\infty^2}{T_w/T_\infty} - 1$$

Substitution of Eqs. (51) and (48) gives

$$\frac{d(\bar{u}/u_\infty)}{\left[1 + B\left(\dfrac{\bar{u}}{u_\infty}\right) - A^2\left(\dfrac{\bar{u}}{u_\infty}\right)^2\right]^{1/2}} = \frac{1}{u_\infty}\cdot\frac{1}{K}\sqrt{\frac{\tau_w}{\rho_w}}\cdot\frac{dy}{y} \tag{52}$$

integration of which yields

$$\frac{1}{A}\sin^{-1}\frac{2A^2\,(\bar{u}/u_\infty) - B}{(B^2 + 4A^2)^{1/2}} = \text{constant} + \frac{1}{u_\infty}\cdot\frac{1}{K}\sqrt{\frac{\tau_w}{\rho_w}}\ln y \tag{53}$$

Since Eq. (53) must reduce to the incompressible fluid case when $M_\infty = 0$ and $T_w/T_\infty = 1$, it will be necessary to write it in the form

$$\frac{1}{A}\sin^{-1}\frac{2A^2(\bar{u}/u_\infty) - B}{(B^2 + 4A^2)^{1/2}} + \frac{1}{A}\sin^{-1}\frac{B}{(B^2 + 4A^2)^{1/2}} = \text{constant} + \frac{1}{u_\infty}\cdot\frac{1}{K}\sqrt{\frac{\tau_w}{\rho_w}}\ln y$$

or

$$\frac{1}{A}\sin^{-1}\frac{2A^2\,(\bar{u}/u_\infty) - B}{(B^2 + 4A^2)^{1/2}} + \frac{1}{A}\sin^{-1}\frac{B}{(B^2 + 4A^2)^{1/2}} = \frac{1}{u_\infty}\sqrt{\frac{\tau_w}{\rho_w}}\left(F + \frac{1}{K}\ln\sqrt{\frac{\tau_w}{\rho_w}}\frac{y}{\nu_w}\right) \tag{54}$$

where F is a constant and ν_w, the kinematic viscosity at the wall, is introduced because of its influence in the laminar sublayer.

Now, the wall shear stress is computed from the formula

$$\tau_w = \frac{d}{dx}\int^{\delta}\bar{\rho}\cdot\bar{u}(u_\infty - \bar{u})dy \tag{55}$$

in which the double and triple correlations have been neglected and where δ is the thickness of the boundary layer. When Eq. (51) and dy obtained from Eq. (54) are substituted into Eq. (55), there results

$$\tau_w = \frac{D\mu_w u_\infty}{K}\cdot\frac{d}{dx}\left\{Ja^2 \exp\left[\frac{a}{A}\sin^{-1}\frac{B}{(B^2+4A^2)^{1/2}}\right]\right\} \tag{56}$$

where

$$D = e^{-FK}$$
$$a = Ku_\infty/\sqrt{\tau_w/\rho_w}$$
$$J = \int_0^1 \frac{z(1-z)}{(1+Bz-A^2z^2)^{3/2}} \times \exp\left[\frac{a}{A}\sin^{-1}\frac{2A^2z-B}{(B^2+4A^2)^{1/2}}\right]dz \tag{57}$$

and $z = \bar{u}/u_\infty$. In these equations, x has its usual meaning of distance along the flat plate in the direction of the free stream and measured from the leading edge.

Eq. (57) for J can be expanded in a series with the aid of integration by parts. The series can be approximated by

$$J = \frac{1}{a^2(1+B-A^2)^{1/2}} \times \exp\left[\frac{a}{A}\sin^{-1}\frac{2A^2-B}{(B^2+4A^2)^{1/2}}\right] + \frac{1}{a^2} \times \exp\left[-\frac{a}{A}\sin^{-1}\frac{B}{(B^2+4A^2)^{1/2}}\right] \tag{58}$$

when terms of higher order than $1/a^2$ are neglected since a is large. However, under usual conditions of heating and cooling, the second term on the right-hand side of Eq. (58) can be dropped compared to the first term, and therefore J can be further approximated by

$$J = \frac{1}{a^2(1+B-A^2)^{1/2}}\exp\left[\frac{a}{A}\sin^{-1}\frac{2A^2-B}{(B^2+4A^2)^{1/2}}\right] \tag{59}$$

Substitution of Eq. (59) into Eq. (56) yields

$$\tau_w = \frac{D\mu_w u_\infty}{K(1+B-A^2)^{1/2}} \times \frac{d}{dx}\left\{\exp\left[\frac{a}{A}\left(\sin^{-1}\frac{2A^2-B}{(B^2+4A^2)^{1/2}} + \sin^{-1}\frac{B}{(B^2+4A^2)^{1/2}}\right)\right]\right\} \tag{60}$$

or, upon rearrangement,

$$\frac{\rho_w u_\infty}{\mu_w}dx = \frac{D}{K^3}\cdot\frac{1}{(1+B-A^2)^{1/2}} \times a^2 d\left\{\exp\left[\frac{a}{A}(\sin^{-1}\alpha + \sin^{-1}\beta)\right]\right\} \tag{61}$$

where

$$\alpha = \frac{2A^2-B}{(B^2+4A^2)^{1/2}} \text{ and } \beta = \frac{B}{(B^2+4A^2)^{1/2}}$$

It is now assumed that the x-variation of the wall temperature is small compared to the x-variation of the wall friction. This is justified in the case of cooling due to radiation because the heat emission due to radiation varies as the fourth power of the wall temperature, whereas, in comparison, the heat transfer from the fluid [see Eq. (40)] varies directly as the wall shear. Certainly, when the boundary layer is insulated, the wall temperature is constant along the plate. If the shear at the leading edge ($x = 0$) of the plate is assumed infinite, the lower limit on a will be zero. Hence, integrating Eq. (61) over x and a and assuming that the upper limit of a is large, there results

$$\frac{\rho_w u_\infty x}{\mu_w} = \frac{a^2}{(1+B-A^2)^{1/2}}\cdot\frac{D}{K^3}\cdot\exp\left[\frac{a}{A}(\sin^{-1}\alpha + \sin^{-1}\beta)\right] \tag{62}$$

or

$$\frac{(2)^{1/2}}{A(c_{fw})^{1/2}}(\sin^{-1}\alpha + \sin^{-1}\beta) = \text{const.} + \frac{1}{K}\ln\left[c_{fw}R_w\left(\frac{T_w}{T_\infty}\right)^{-1/2}\right] \tag{63}$$

where

$$\frac{T_w}{T_\infty} = \left(1 + \frac{\gamma-1}{2}M_\infty^2\right)^{-1}, \quad c_{fw} = \frac{2\tau_w}{\rho_w u_\infty^2} = \frac{2K^2}{a^2}$$
$$R_w = \frac{\rho_w u_\infty x}{\mu_w}$$

Eq. (63) can be transformed into an equation in terms of the free-stream rather than wall conditions. In the first place, since $\rho_w = \rho_\infty(T_\infty/T_w)$, it follows that $c_{fw} = c_{f_\infty}(T_w/T_\infty)$. Secondly, a power law for viscosity can be assumed [namely, $\mu_w = \mu_\infty(T_w/T_\infty)^\omega$] whence $R_w = R_\infty(T_w/T_\infty)^{-(1+\omega)}$. Eq. (63) then becomes

$$\frac{(2)^{1/2}}{A(c_{f_\infty})^{1/2}(T_w/T_\infty)^{1/2}}(\sin^{-1}\alpha + \sin^{-1}\beta) = \text{const.} + \frac{1}{K}\left(\ln R_\infty c_{f_\infty} - \frac{1+2\omega}{2}\ln\frac{T_w}{T_\infty}\right) \tag{64}$$

The constant in Eq. (64) can be determined from the requirement that, when $M_\infty = 0$ and $T_w/T_\infty = 1$, Eq. (64) must reduce to the incompressible fluid case[7]—namely,

$$1/(c_{f_\infty})^{1/2} = 1.70 + 4.15\log_{10}R_\infty c_{f_\infty} \tag{65}$$

for local skin friction. Adjustment of the constant yields the following final equation for local skin-friction coefficient when wall temperature, free-stream Reynolds Number, and Mach Number are arbitrary:

$$\frac{0.242}{A(c_{f_\infty})^{1/2}(T_w/T_\infty)^{1/2}}(\sin^{-1}\alpha + \sin^{-1}\beta) = 0.41 + \log_{10}R_\infty c_{f_\infty} - \frac{1+2\omega}{2}\log_{10}\frac{T_w}{T_\infty} \tag{66}$$

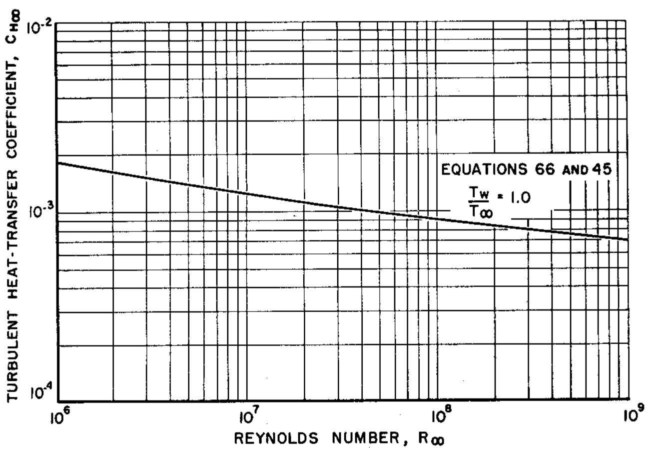

FIG. 1. Turbulent heat-transfer coefficient for air at $M_\infty = 0$.

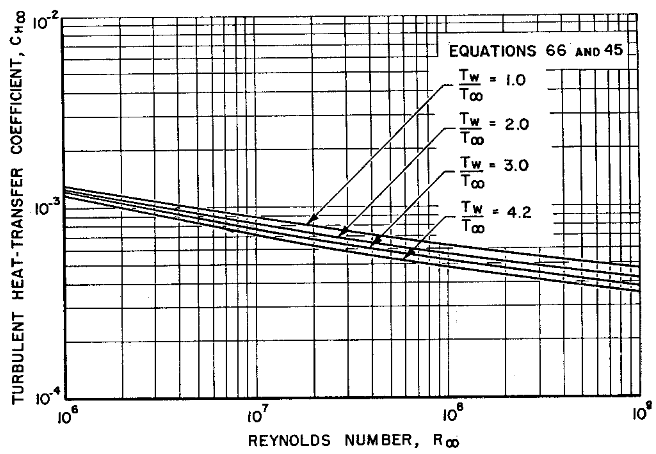

FIG. 3. Turbulent heat-transfer coefficient for air at $M_\infty = 4$.

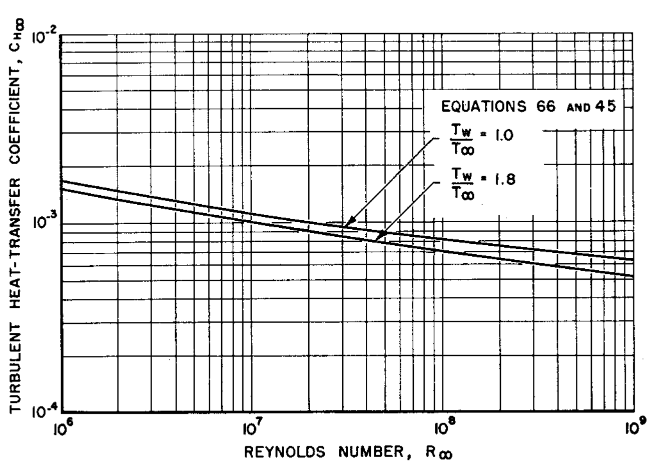

FIG. 2. Turbulent heat-transfer coefficient for air at $M_\infty = 2$.

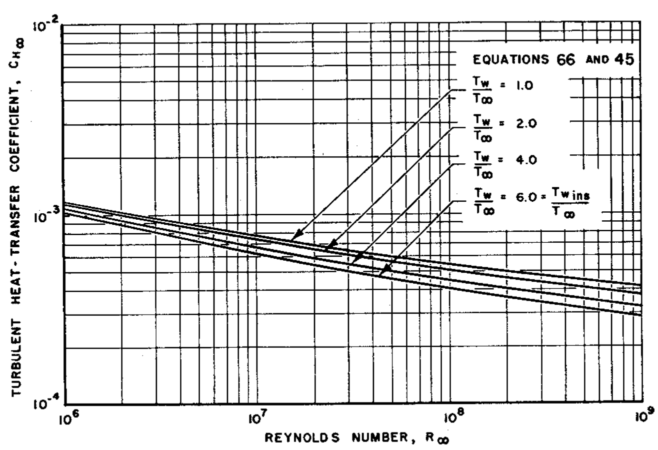

FIG. 4. Turbulent heat-transfer coefficient for air at $M_\infty = 5$.

The local heat transfer can now be obtained by using Eq. (45).

Figs. 1 to 7 present values of the turbulent heat-transfer coefficient computed from Eqs. (66) and (45) for Mach Numbers 0, 2, 4, 5, 6, 8, and 10 and various temperature ratios. In these figures, the highest value of T_w/T_∞ corresponds to the insulated case. Values used for ω and γ are 0.76 and 1.400, respectively.

A formula for the total (mean) skin-friction coefficient can also be derived. In terms of wall condition, the mean friction coefficient is defined by

$$\frac{1}{2} C_{fw}\rho_w u_\infty{}^2 x = \int_0^x \tau_w dx$$

Direct integration of Eq. (60) then yields, for large values of a,

$$\frac{1}{2} C_{fw}R_w = \left(\frac{T_w}{T_\infty}\right)^{1/2} \frac{D}{K} \exp\left[\frac{a}{A}\left(\sin^{-1}\alpha + \sin^{-1}\beta\right)\right] \quad (67)$$

Comparison of Eqs. (67) and (62) requires that $C_{fw} \rightarrow c_{fw}$ at least for large values of a—that is, for small values of c_{fw}. Bearing this in mind and knowing that the local and mean frictions are different at moderately large friction coefficients, the form of the equation for

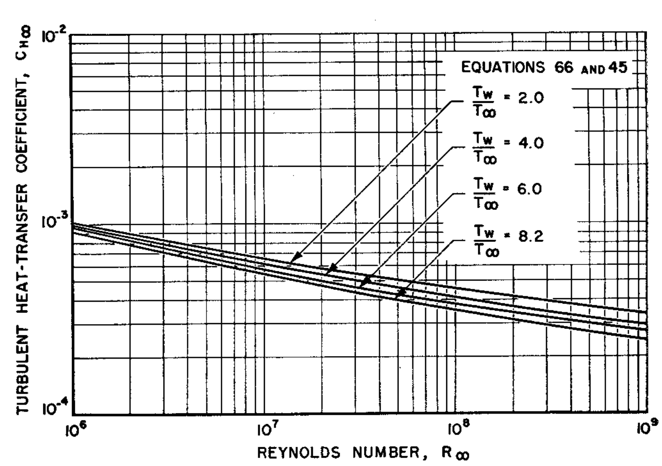

FIG. 5. Turbulent heat-transfer coefficient for air at $M_\infty = 6$.

C_{fw} is maintained the same as that of Eq. (63), although the constant is different. Hence,

$$\frac{2^{1/2}}{A(C_{fw})^{1/2}}\left(\sin^{-1}\alpha + \sin^{-1}\beta\right) = \text{const.} + \frac{1}{K}\left[\ln C_{fw}R_w \left(\frac{T_w}{T_\infty}\right)^{-1/2}\right] \quad (68)$$

or, in terms of free-stream conditions and with the use of the power viscosity law,

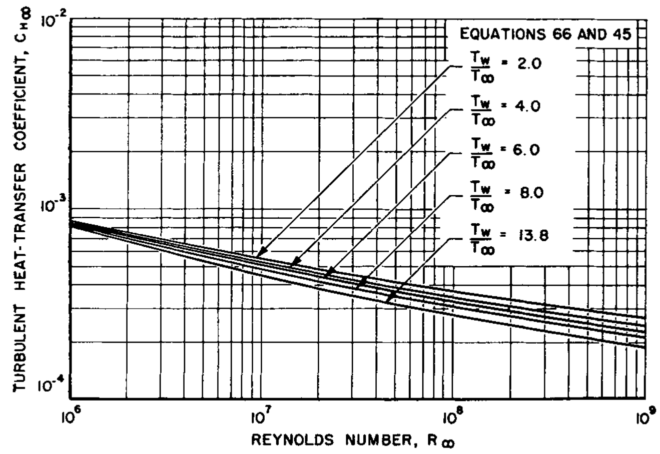

FIG. 6. Turbulent heat-transfer coefficient for air at $M_\infty = 8$.

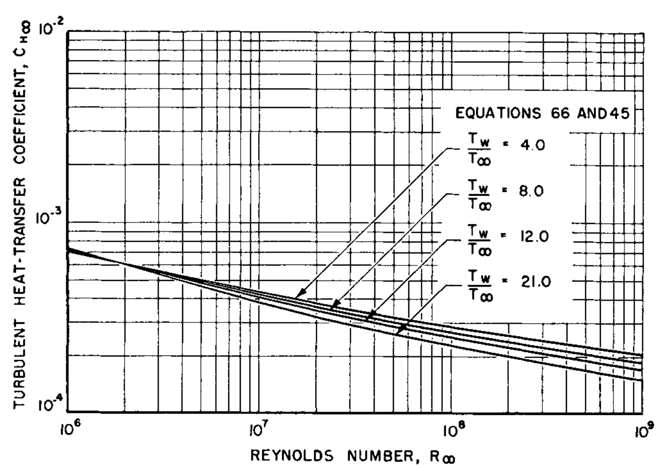

FIG. 7. Turbulent heat-transfer coefficient for air at $M_\infty = 10$.

$$\frac{2^{1/2}}{A(C_{f_\infty})^{1/2}(T_w/T_\infty)^{1/2}}(\sin^{-1}\alpha + \sin^{-1}\beta) = \text{const.} + \frac{1}{K}\left(\ln R_\infty C_{f_\infty} - \frac{1+2\omega}{2}\ln\frac{T_w}{T_\infty}\right) \quad (69)$$

The constant in Eq. (69) can be determined, in a manner similar to that above, when the requirement is made that Eq. (69) must reduce to the von Kármán incompressible, mean skin-friction law—namely,

$$0.242/C_{f_\infty}{}^{1/2} = \log_{10}R_\infty C_{f_\infty} \quad (70)$$

when $M_\infty = 0$ and $T_w/T_\infty = 1$. Hence, the final formula for mean skin friction is

$$\frac{0.242}{A(C_{f_\infty})^{1/2}(T_w/T_\infty)^{1/2}}(\sin^{-1}\alpha + \sin^{-1}\beta) = \log_{10}R_\infty C_{f_\infty} - \frac{1+2\omega}{2}\log_{10}\frac{T_w}{T_\infty} \quad (71)$$

Figs. 8 to 14 present values of the turbulent mean skin-friction coefficient obtained from Eq. (71) for Mach Numbers 0, 2, 4, 5, 6, 8, and 10 and various temperature ratios. In these figures, the highest values of T_w/T_∞ correspond to the insulated case. The values used for ω and γ are 0.76 and 1.400, respectively.

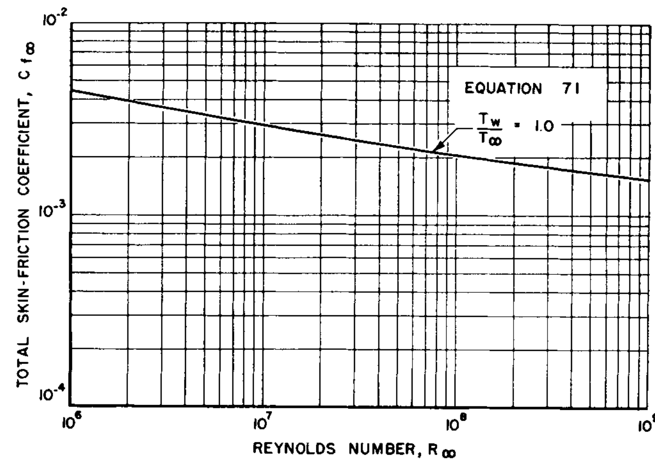

FIG. 8. Total skin-friction coefficient for air at $M_\infty = 0$.

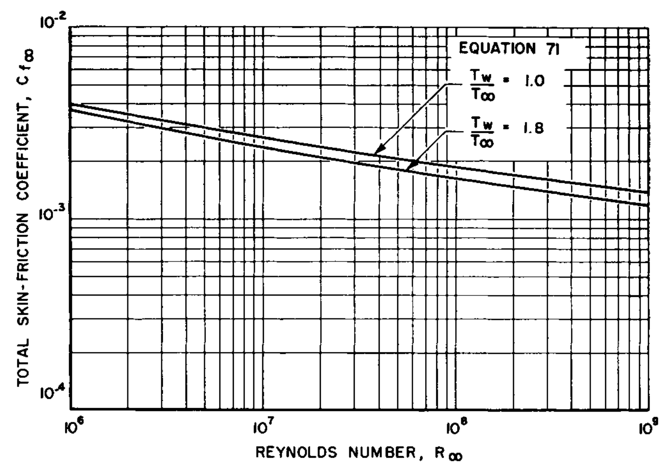

FIG. 9. Total skin-friction coefficient for air at $M_\infty = 2$.

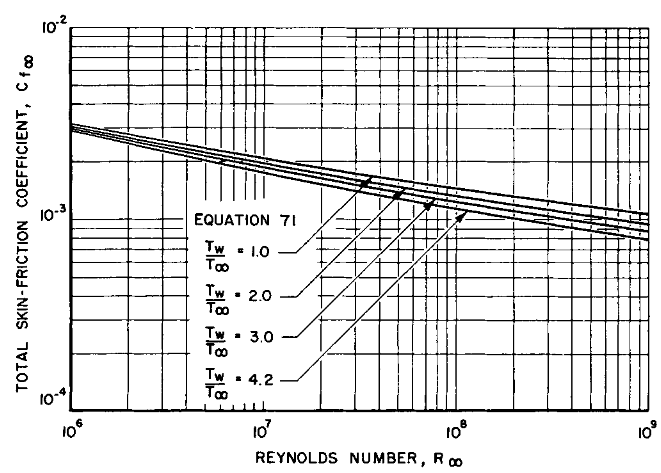

FIG. 10. Total skin-friction coefficient for air at $M_\infty = 4$.

As a matter of interest, the temperature through the boundary layer has been computed for $M_\infty = 5$ and $R_\infty = 10^7$ for various wall temperatures using Eqs. (34) and (54) in the form

$$\sin^{-1}\frac{A(u/u_\infty) - (B/2A)}{[(B/2A)^2 + 1]^{1/2}} - \sin^{-1}\frac{A - (B/2A)}{[(B/2A)^2 + 1]^{1/2}} = 5.75\left(\frac{\gamma - 1}{2}M_\infty{}^2\right)^{1/2}\left(\frac{c_{f_\infty}}{2}\right)^{1/2}\log_{10}\frac{y}{\delta} \quad (72)$$

where δ is the boundary-layer thickness and K is taken as 0.40. When the laminar sublayer is disregarded, the temperature distribution appears as shown in Figs. 15 and 16.

A relationship between the local and mean friction coefficients can be found in the following manner: Since

$$\frac{1}{2}C_{fw}\rho_w u_\infty^2 x = \int_0^x \tau_w dx = \frac{1}{2}\rho_w u_\infty^2 \int_0^x c_f\, dx$$

it follows that

$$c_{fw} = \frac{d(C_{fw}x)}{dx} = C_{fw} + x\frac{dC_{fw}}{dx} \qquad (73)$$

Differentiation of Eq. (68) with respect to x gives

$$-\frac{\sqrt{2}}{2}\cdot\frac{K}{A}\cdot(\sin^{-1}\alpha + \sin^{-1}\beta)\, C_{fw}^{-3/2}\frac{dC_{fw}}{dx} = \frac{1}{C_{fw}R_w}\frac{d(C_{fw}R_w)}{dx}$$

or, upon multiplication of both sides by x and expansion of the right-hand side,

$$-\frac{\sqrt{2}}{2}\cdot\frac{K}{A}(\sin^{-1}\alpha + \sin^{-1}\beta)\, C_{fw}^{-3/2}\cdot x\cdot\frac{dC_{fw}}{dx} = \frac{1}{R_wC_{fw}}\left(R_w\cdot x\cdot\frac{dC_{fw}}{dx} + C_{fw}\cdot x\cdot\frac{dR_w}{dx}\right) \qquad (74)$$

But $R_w = \rho_w u_\infty x/\mu_w$, whence $x(dR_w/dx) = R_w$, so that Eq. (74) yields, after collection of terms,

$$x\frac{dC_{fw}}{dx} = -\left[\frac{\sqrt{2}}{2}\cdot\frac{K}{A}(\sin^{-1}\alpha + \sin^{-1}\beta)\, C_{fw}^{-3/2} + C_{fw}^{-1}\right]^{-1}$$

This expression substituted into Eq. (73) gives

$$c_{fw} = \frac{0.558\,(1/A)\,(\sin^{-1}\alpha + \sin^{-1}\beta)C_{fw}}{0.558\,(1/A)\,(\sin^{-1}\alpha + \sin^{-1}\beta) + 2C_{fw}^{1/2}} \qquad (75)$$

where $\sqrt{2}K = 0.558$. Again, $C_{fw} = C_{f_\infty}(T_w/T_\infty)$ and $c_{fw} = c_{f_\infty}(T_w/T_\infty)$ so that Eq. (75) becomes

$$c_{f_\infty} = \frac{0.558\,(1/A)\,(\sin^{-1}\alpha + \sin^{-1}\beta)C_{f_\infty}}{0.558\,\frac{1}{A}\,(\sin^{-1}\alpha + \sin^{-1}\beta) + 2C_{f_\infty}^{1/2}\left(\frac{T_w}{T_\infty}\right)^{1/2}} \qquad (76)$$

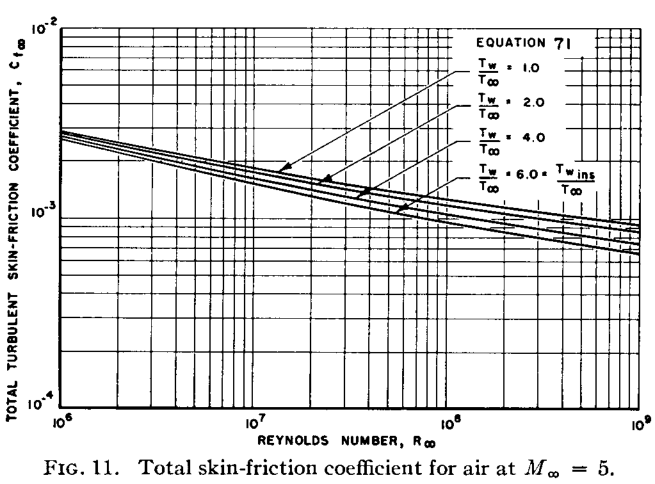

FIG. 11. Total skin-friction coefficient for air at $M_\infty = 5$.

Turbulent Skin Friction on an Insulated Plate

When the heat transfer to or from the boundary layer is zero, the wall temperature is given by

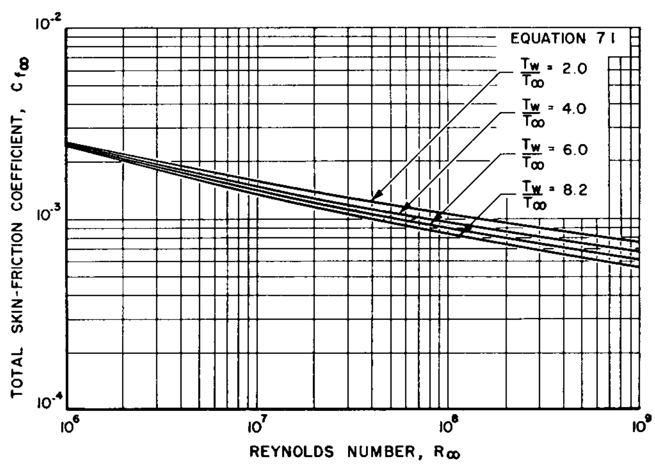

FIG. 12. Total skin-friction coefficient for air at $M_\infty = 6$.

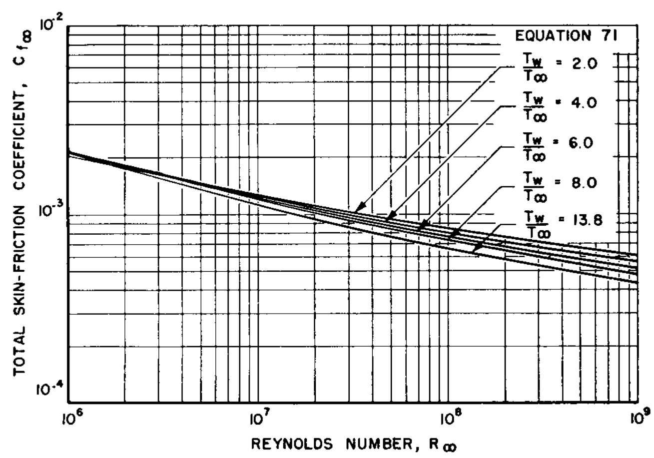

FIG. 13. Total skin-friction coefficient for air at $M_\infty = 8$.

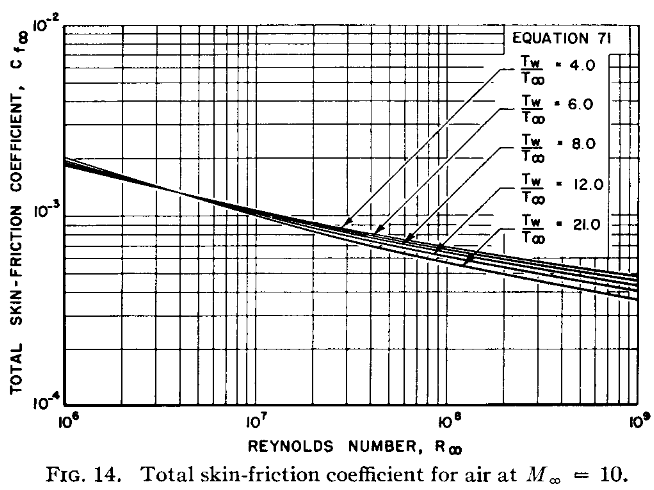

FIG. 14. Total skin-friction coefficient for air at $M_\infty = 10$.

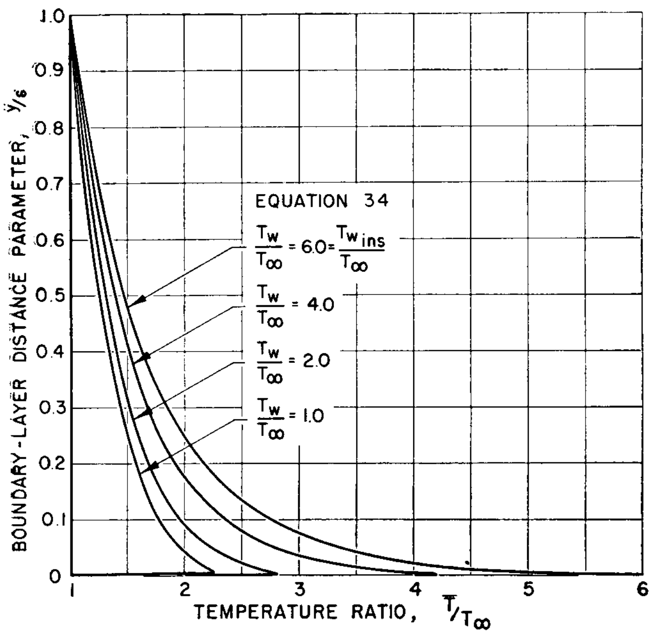

FIG. 15. Temperature distribution through the turbulent boundary layer for air at $R_\infty = 10^7$ and $M_\infty = 5$.

$$(T_w/T_\infty) = 1 + [(\gamma - 1)/2]M_\infty^2$$

The mean skin-friction formula [Eq. (71)] becomes, then,

$$\frac{0.242}{C_{f_\infty}^{1/2}}(1 - \lambda^2)^{1/2}\frac{\sin^{-1}\lambda}{\lambda} = \log_{10} R_\infty C_{f_\infty} + \frac{1 + 2\omega}{2}\log_{10}(1 - \lambda^2) \quad (77)$$

where

$$1 - \lambda^2 = \{1 + [(\gamma - 1)/2]M_\infty^2\}^{-1}$$

Eq. (77) is plotted in Fig. 18 for $\omega = 0.76$ and $\gamma = 1.4$. Also plotted are data obtained at Daingerfield.[8]

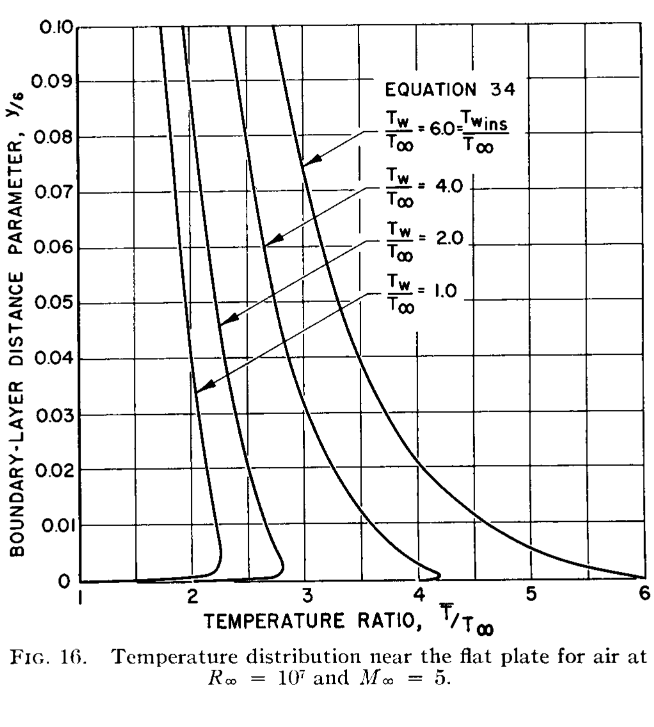

FIG. 16. Temperature distribution near the flat plate for air at $R_\infty = 10^7$ and $M_\infty = 5$.

It appears that the data are approaching the writer's theoretical curves.

The Sutherland formula for viscosity—namely,

$$\frac{\mu}{\mu_\infty} = \left(\frac{T}{T_\infty}\right)^{3/2}\frac{T_\infty + S}{T + S} \quad (78)$$

can also be used. In this formula T is absolute temperature and S is a constant. According to the data of Tribus and Boelter[9] (see Fig. 17), it is found that $S = 110°K$. Use of this formula in Eq. (68) ultimately yields, for the insulated-plate case,

$$\frac{0.242}{C_{f_\infty}^{1/2}}(1 - \lambda^2)^{1/2}\frac{\sin^{-1}\lambda}{\lambda} = \log_{10} R_\infty C_{f_\infty} + \log_{10}(1 - \lambda^2)\left(1 - \frac{\theta\lambda^2}{1 + \theta}\right) \quad (79)$$

where $\theta = S/T_\infty$.

The ratio of the compressible to incompressible fluid skin-friction coefficient, computed from Eq. (77) at Reynolds Number of 10^7, is plotted in Fig. 19. For the purpose of comparing results obtained using the power viscosity law and the more accurate Sutherland law, this ratio was also computed from Eq. (79) at the same Reynolds Number and is plotted in Fig. 19. The free-stream temperature was taken as −67.6°F. It is seen that the difference is not great, indeed practically negligible. Included in the figure is the friction variation for laminar boundary layers.

von Kármán's results for turbulent skin friction are also plotted in Fig. 19. These results are obtained by substituting wall conditions directly into Eq. (70) to produce

$$\frac{0.242}{C_{f_\infty}^{1/2}}(1 - \lambda^2)^{1/2} = \log_{10}R_\infty C_{f_\infty} + \omega \log_{10}(1 - \lambda^2) \quad (80)$$

using the power viscosity law. The reason for the difference between the writer's results and those of von Kármán is simply that von Kármán has assumed that the wall conditions apply all the way across the boundary layer, whereas the writer has allowed the density to vary with velocity in the case of constancy of energy per unit mass throughout the layer. It seems that von Kármán's formula would be optimistic from the point of view of drag.

It is now advisable to reiterate how von Kármán's wall procedure was developed. For lack of more general information, von Kármán[3] used Busemann's[10] discovery (that for a high Mach Number the velocity profile in the case of laminar flow is approximately linear) and arrived at an approximation for the variation of laminar friction with Mach Number (see Fig. 20). von Kármán then said that this result could be approximated by merely replacing the free-stream conditions of density and viscosity by the wall condition in the Blasius formula for laminar incompressible flow. Hence, given

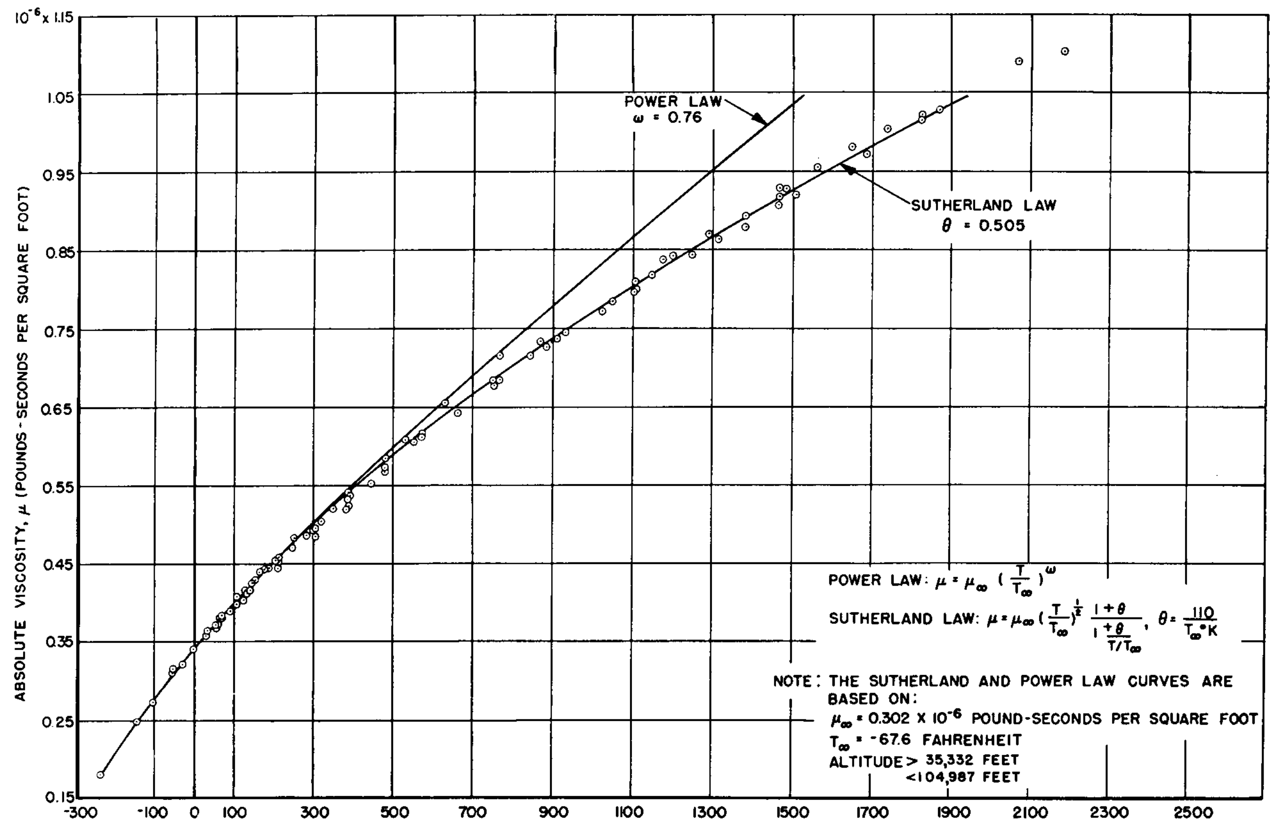

FIG. 17. Comparison of the power and Sutherland viscosity laws with experimental data.

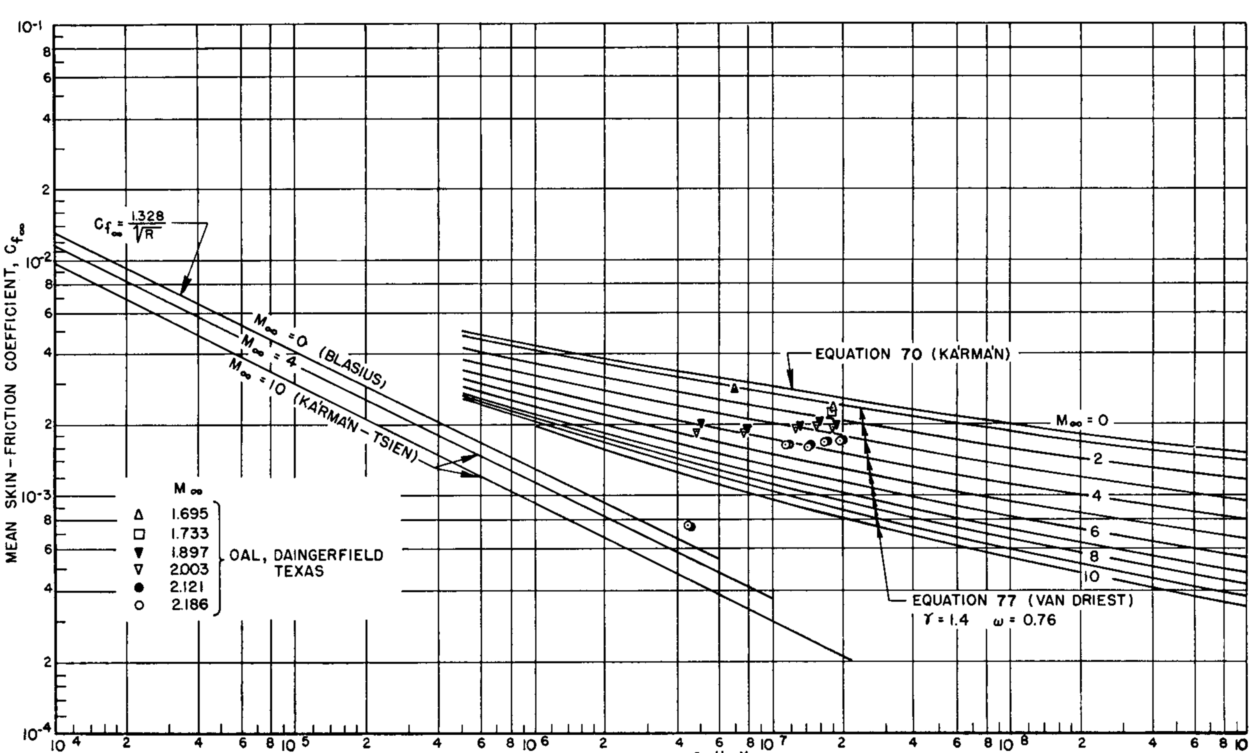

FIG. 18. Mean turbulent skin friction as a function of Reynolds Number and Mach Number for zero heat transfer.

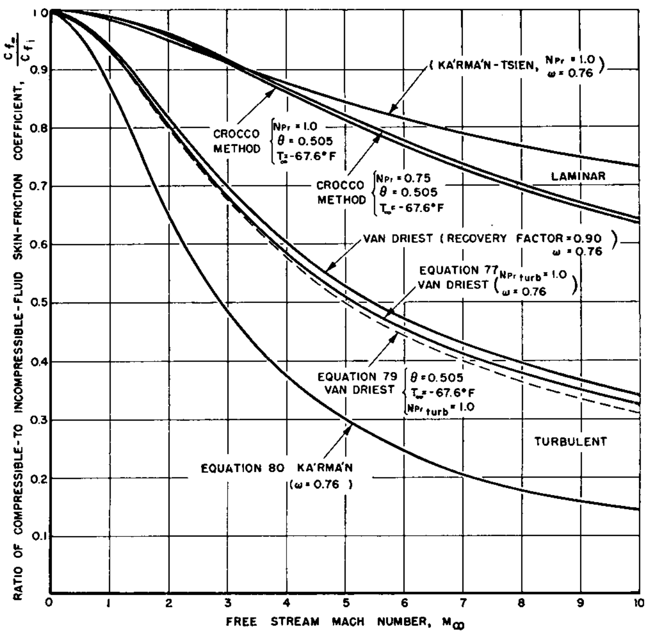

FIG. 19. Comparison of different laws for compressible-flow skin friction at $R_\infty = 10^7$.

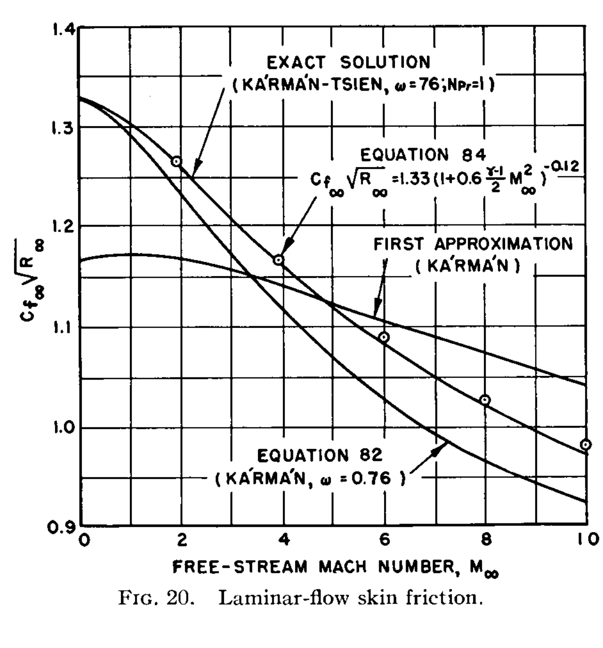

FIG. 20. Laminar-flow skin friction.

$$C_{f_\infty}\sqrt{\rho_\infty u_\infty x/\mu_\infty} = 1.33 \quad (81)$$

and $R_w = R_\infty(1 - \lambda^2)^{1+\omega}$ with $C_{fw} = C_{f_\infty}(1 - \lambda^2)^{-1}$, there results

$$C_{f_\infty}\sqrt{\frac{\rho_\infty u_\infty x}{\mu_\infty}} = 1.33\,(1 - \lambda^2)^{(1-\omega)/2} = 1.33\left(1 + \frac{\gamma - 1}{2} M_\infty^2\right)^{(\omega-1)/2} \quad (82)$$

which, for $\omega = 0.76$, does cross the linear approximation at $M_\infty = 3.5$ (see Fig. 20). The fact that the linear approximation could be reapproximated by using wall conditions in the incompressible flow formula gave von Kármán the idea that perhaps a first approximation for the turbulent compressible case of skin friction could be obtained by using wall conditions in the turbulent incompressible formula. As a result, he obtained Eq. (80).

That the use of wall conditions is not the correct procedure is seen in Fig. 20, where the exact von Kármán-Tsien solution for $\omega = 0.76$ and $N_{Pr} = 1$ is plotted. However, if a portion of the temperature rise—namely,

$$C = (\bar{T} - T_\infty)/(T_w - T_\infty) \quad (83)$$

is used rather than the full rise to wall temperature, Eq. (82) can be corrected to yield

$$C_{f_\infty}\sqrt{\frac{\rho_\infty u_\infty x}{\mu_\infty}} = 1.33\left(1 + C\frac{\gamma - 1}{2} M_\infty^2\right)^{(\omega-1)/2} \quad (84)$$

It will be found that a value of $C = 0.60$ duplicates exactly the theoretical curve of von Kármán and Tsien.

Likewise, it can be shown that the wall conditions are not satisfactory for use in computing the laminar boundary-layer thickness from the incompressible formula

$$\delta\sqrt{\rho_\infty u_\infty/\mu_\infty x} = 5.2 \quad (85)$$

A value of $C = 0.45$ yields the formula

$$\delta\sqrt{\frac{\rho_\infty u_\infty}{\mu_\infty x}} = 5.2\left(1 + 0.45\frac{\gamma - 1}{2} M_\infty^2\right)^{(\omega+1)/2} \quad (86)$$

which fits the values obtained by von Kármán and Tsien (see Fig. 21).

The proper temperature to be used in the incompressible fluid skin-friction formula in order to obtain results given by Eq. (77) can be computed. For $M_\infty = 3$ and $R_\infty = 10^7$, the value of C [Eq. (83)] is equal to 0.41.

An estimate of the effect of Mach Number on the growth of the turbulent boundary layer can be made. Assuming that the turbulent flow starts from the leading edge of the plate, the thickness δ_c of the turbulent compressible fluid boundary layer is given by

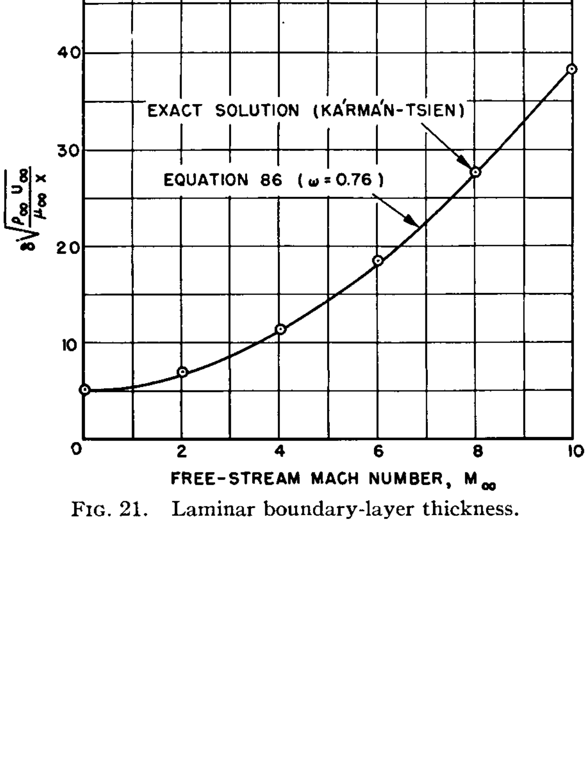

FIG. 21. Laminar boundary-layer thickness.

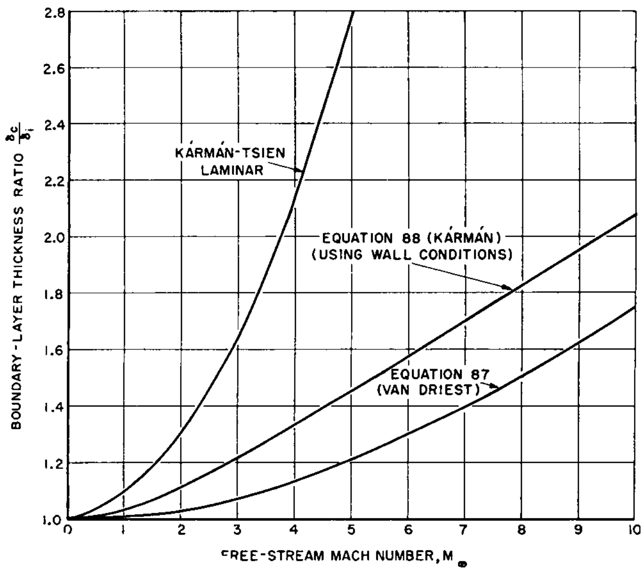

FIG. 22. Comparison of different laws for compressible-flow boundary-layer thickness at $R_\infty = 10^7$.

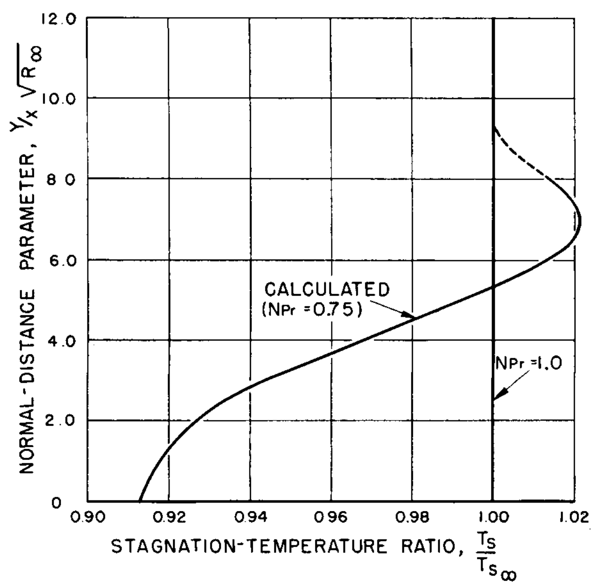

FIG. 23. Stagnation-temperature ratio variation across the laminar boundary layer at $M_\infty = 3$.

$$\delta_c = \frac{1}{2} C_{f_\infty} \cdot x \cdot \frac{1}{\int_0^1 \frac{\bar{\rho}}{\rho_\infty} \cdot \frac{\bar{u}}{u_\infty}\left(1 - \frac{\bar{u}}{u_\infty}\right) d\left(\frac{y}{\delta}\right)} \tag{87}$$

since

$$\int_0^x \tau_w dx = \int_0 \bar{\rho}\bar{u}(u_\infty - \bar{u})dy$$
$$= \rho_\infty u_\infty^2 \delta \int_0^1 \frac{\bar{\rho}}{\rho_\infty} \cdot \frac{\bar{u}}{u_\infty}\left(1 - \frac{\bar{u}}{u_\infty}\right) d\left(\frac{y}{\delta}\right)$$

and

$$C_{f_\infty} \frac{\rho_\infty u_\infty^2 x}{2} = \int_0^x \tau_w \, dx$$

Eq. (87) is plotted in Fig. 22 at $R_\infty = 10^7$, assuming that Eq. (72) represents the velocity profile and using the planimeter to obtain the integral. The von Kármán-Tsien laminar thickness variation is likewise shown.

A turbulent compressible fluid boundary-layer thickness can also be computed using wall conditions in the formula for the incompressible fluid thickness δ_i. If the logarithmic velocity profile is assumed to hold, δ_i can be obtained in closed form because the integral [Eq. (87)] can be readily evaluated when $M_\infty = 0$; the result is

$$\delta_i = \frac{1}{2} C_f \cdot x \cdot \frac{0.558}{c_f^{1/2}[1 - (2c_f^{1/2}/0.558)]} \tag{88}$$

where, from Eq. (76),

$$c_f = 0.558C_f/(0.558 + 2C_f^{1/2}) \tag{89}$$

Values of C_{f_∞}, obtained from Eq. (70) when the wall Reynolds Numbers are used, are then substituted into Eq. (88) and (89) to yield values of δ_c. The relative thickness obtained in this manner is also plotted in Fig. 22. The constant C [defined by Eq. (83)], which is required to make the incompressible-flow formula [Eq. (88)] yield the compressible-flow thickness [given by Eq. (87)], is 0.24 at $M_\infty = 3$.

Discussion of the Theory

Recovery Factor

It was assumed in the development of the foregoing theory that the turbulence Prandtl Number $c_p\epsilon/\kappa$, as well as the laminar Prandtl Number $c_p\mu/k$, was equal to unity. As a result of this assumption, it was concluded that in the case of the insulated plate the total energy per unit mass, defined by $c_p\bar{T} + (\bar{u}^2/2)$, is constant across the turbulent boundary layer. This is the same conclusion reached for fully laminar layers when the Prandtl Number is assumed equal to unity.

Now, in the case of the insulated laminar layer, the fact that the Prandtl Number is not unity, but rather about 0.75, results in a total energy distribution similar to the one shown in Fig. 23 in which the energy has migrated from regions near the wall to regions near the free stream. The curve of Fig. 23 has been computed[11] using the Crocco method to solve the differential equations. A manifestation of this migration is the usual experimentally observed wall temperature of insulated plates which is lower than the total free-stream temperature. This experimental fact is specified by the so-called recovery factor, defined by $\eta = (T_{w_{ins.}} - T_\infty)/(T_{\infty_T} - T_\infty)$, in which T_{∞_T} is equal to $T_\infty\{1 + [(\gamma - 1)/2]M_\infty^2\}$. With laminar boundary layers, the recovery factor, $\eta_{lam.}$, is approximately equal to $(N_{Pr})^{1/2}$.

However, in the case of the insulated fully turbulent boundary layer, one would expect that the turbulence Prandtl Number would also be somewhat different from unity. Indeed, a recovery factor different from unity

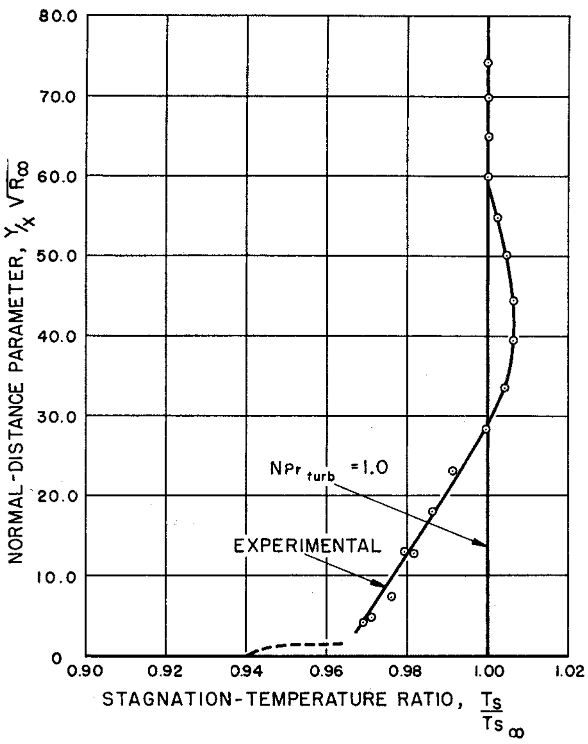

FIG. 24. Stagnation-temperature ratio variation across the turbulent boundary layer at $M_\infty = 2.8$.

is observed with turbulent layers. It is found[12] that $\eta_{turb.}$ is approximately given by $(N_{Pr_{lam.}})^{1/3}$ Furthermore, total temperature measurements[13] have been carried out at the Aerophysics Laboratory of North American Aviation, Inc. (see Fig. 24). These measurements were made in a $^1/_4$-in. turbulent boundary layer on the wall of the N.A.A. 50 sq.cm. supersonic wind tunnel. Fig. 24 definitely shows a pattern similar to that of Fig. 23. It appears that the assumption of Prandtl Number equal to unity is a better approximation for the turbulent case than for the laminar.

The error involved in skin-friction calculation because of the assumption of Prandtl Number equal to unity can be estimated in the following manner: It is first noted that in the case of insulated laminar layers the temperature rise for $N_{Pr} = 0.75$ is proportional to the temperature rise for $N_{Pr} = 1$ at points throughout the boundary layer. It is then assumed that in the case of insulated turbulent layers the effect of recovery factor on the temperature distribution is roughly the same. This leads to the conclusion that an effective Mach Number rather than the true Mach Number should be used in the above-developed friction formula. Since $T_{w_{ins.}} = T_\infty \{1 + \eta[(\gamma - 1)/2]M_\infty^2\}$ and $\eta_{turb.} \approx 0.9$, it follows that the effective Mach Number should be about $0.95\ M_\infty$, all other properties remaining unchanged. The effect of recovery factor on C_f would be negligible as seen in Fig. 19. This is compatible with the same conclusion reached for laminar layers (Fig. 19). Because of the recovery factor for turbulent layers, the equation for heat transfer would become

$$q_w = C_{H_\infty} c_p \rho_\infty u_\infty T_\infty \left[1 + \frac{\gamma - 1}{2}(0.95\ M_\infty)^2 - \frac{T_w}{T_\infty}\right] \tag{90}$$

in which, as previously stated, a more exact value of C_H than that given by Eq. (45) requires a more exact knowledge of the turbulence mechanism.

Velocity Distribution

The turbulent velocity distribution computed by the foregoing theory [Eq. (72)] is expected to be too full. This is concluded from the fact that, in the case of incompressible flow measurements[14] on flat plates with zero pressure gradients, the velocity distribution seems to follow a power law rather than a semilogarithmic law. The difference between the law for plates with no pressure gradient and the semilogarithmic law for pipes with pressure gradients is apparently due to the existence or nonexistence of the pressure gradient. (Certainly, in the case of laminar flow, the parabolic velocity distribution in a pipe is fuller than in the Blasius solution for the plate.) However, as far as the derivation of a law for incompressible fluid skin friction is concerned, use of the logarithmic law for velocity yields a logarithmic friction law that contains the pertinent variables in the proper functional form but whose constants must be determined by experiment. In the development of the above theory for compressible fluids, the same procedure is assumed valid.

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